THE SOLUTION BY ITERATION OF NONLINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES[1]

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Introduction. Let X be a Banach space, T a (possibly) nonlinear mapping of X into X. We are concerned with the solvability of the equation

$$u - Tu = f \tag{1}$$

for a given element f of X and its relation to the properties of the Picard iterates for the Equation (1), i.e. the sequence $\{x_n\}$ where

$$x_{n+1} = Tx_n + f, \quad x_0 \text{ given.} \tag{2}$$

In a preceding note on the linear case [8], we established the following facts for linear T:

(a) If X is reflexive and T is asymptotically bounded (i.e. $\|T^n\| \leqq M$ for some constant M and all $n \geqq 1$), then the Equation (1) has a solution u for a given f if and only if for any specific x_0, the sequence of Picard iterates $\{x_n\}$ starting with x_0 is bounded in X (see [2]).

(b) For a general Banach space X, if T is asymptotically convergent (i.e. $T^n x$ converges strongly in X for each x in X as $n \to +\infty$), the sequence of Picard iterates $\{x_n\}$ for a given x_0 converges if and only if the equation (1) has a solution.

(c) For a general Banach space X and T asymptotically convergent, if an infinite subsequence of the sequence $\{x_n\}$ converges, then the whole sequence converges to a solution of Equation (1).

Our object in the present note is to give some partial extensions of these results to a general class of nonlinear operators T, and to indicate some interesting examples of the application of these nonlinear results.

THEOREM 1. *Let T be a nonexpansive nonlinear mapping of X into X, (i.e. $\|Tx-Ty\| \leqq \|x-y\|$ for all x and y in X), and suppose that X is uniformly convex. Then the Equation* (1) *has a solution u for a given f in X if and only if for any specific x_0 in X, the sequence of Picard iterates $\{x_n\}$ starting at x_0 is bounded in X.*

PROOF OF THEOREM 1. Let T_f be the mapping of X into X given by $T_f(u) = Tu + f$. Then u is a solution of Equation (1) if and only if

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u is a fixed point of T_f, and T_f like T is a nonexpansive self-mapping of X. If T_f has such a fixed ponit u, then for each $n \geqq 1$,

$$\|x_{n+1} - u\| = \|T_f(x_n) - T_f(u)\| \leqq \|x_n - u\|. \tag{3}$$

Hence the sequence $\{x_n\}$ is bounded. The converse is a corollary, due to Belluce and Kirk [1], of the result established independently by Browder [6] and Kirk [9] that every nonexpansive self-mapping of a nonempty bounded close convex subset C of a uniformly convex space has a fixed point. Indeed, let d be the diameter of the set x_n, and for each x in X, let $D_d(x)$ be the closed ball of radius d about x. If $C_k = \bigcap_{j \geqq k} D_d(x_j)$, C_k is nonempty and convex for each k, and $T_f(C_k) \subset C_{k+1}$. Let C be the closure of the union of C_k for $k \geqq 1$. Since C_k increases with k, C is a closed bounded convex subset of X. Since T_f maps C into C, T_f has a fixed point in C. q.e.d.

DEFINITION 1. *The mapping T is said to be asymptotically regular if for each x in X, $T^{n+1}x - T^n x \to 0$ strongly in X as $n \to +\infty$. T is said to be weakly asymptotically regular if $T^{n+1}x - T^n x \to 0$ weakly in X as $n \to +\infty$ for each x in X.*

THEOREM 2. *Let X be a Banach space, T a nonexpansive mapping of X into X. For a given f in X, let $T_f(u) = T(u) + f$, and suppose that the mapping T_f is weakly asymptotically regular. Let $x_n = T_f^n x_0$ be the sequence of Picard iterates for the Equation* (1) *starting with x_0, and suppose that an infinite subsequence of the sequence $\{x_n\}$ converges strongly to an element y of X.*

Then y is a solution of Equation (1) *and the whole sequence $\{x_n\}$ converges strongly to y.*

PROOF OF THEOREM 2. If u is a solution of equation (1), i.e. a fixed point of T_f, then by Equation (3) above

$$\|x_{n+1} - u\| \leqq \|x_n - u\|.$$

If an infinite subsequence of $\{x_n\}$ converges to u, it follows that the whole sequence converges to u. Hence it suffices to show that the limit y of the convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is indeed a fixed point of T_f.

By the assumption of weak asymptotic regularity of T_f, however, we know that

$$(I - T_f)(x_n) = (I - T_f)(T_f^n x_0) \to 0$$

weakly in X as $n \to +\infty$. Since $x_{n_k} \to y$ strongly in X and $(I - T_f)$ is continuous from X to X in the strong topology, $(I - T_f)(x_{n_k}) \to (I - T_f)(y)$ strongly in X. Hence $(I - T_f)(y) = 0$. q.e.d.

DEFINITION 2. *The mapping S of X into X is said to be demiclosed if for any sequence $\{u_n\}$ in X with $u_n \to u$ weakly in X, $Su_n \to v$ strongly in X, we have $Su = v$.*

THEOREM 3. *Let X be a Banach space, T a nonexpansive mapping of X into X such that for a given f in X, T_f is asymptotically regular and $(I - T_f)$ is demiclosed. Let F be the set of fixed points of T_f, and $\{x_n\}$ the sequence of Picard iterates for T_f starting at x_0. Suppose that T_f has at least one fixed point.*

Then the weak limit of any weakly convergent subsequence of $\{x_n\}$ lies in F. In particular, if X is reflexive and F consists of exactly one point y, $\{x_n\}$ converges weakly to y.

PROOF OF THEOREM 3. Suppose F is nonempty and let $\{x_{n_j}\}$ be a weakly convergent subsequence of $\{x_n\}$ with weak limit u. By the asymptotic regularity of T_f, $(I - T_f)(x_{n_j}) \to 0$ strongly in X. Since T_f is demiclosed by hypothesis, it follows that $(I - T_f)u = 0$, i.e. u lies in F.

If X is reflexive, each infinite subsequence of $\{x_n\}$ contains a weakly convergent subsequence whose limit lies in F. If F consists of a single point, it follows that x_n converges to that point weakly in X. q.e.d.

THEOREM 4. *Let H be a Hilbert space, T a nonexpansive self-mapping of H such that T_f is asymptotically regular for a given f in H and has a nonempty fixed point set F. Then the weak limit of any weakly convergent subsequence of $\{x_n\}$ lies in F. In particular, if F consists of a single point u, then x_n converges weakly to u.*

More generally, these conclusions are valid for any Banach space X having a weakly continuous duality mapping ([7]).

PROOF OF THEOREM 4. It has been shown in Browder [4] using the theory of monotone operators in Hilbert space that if T_f is a nonexpansive mapping, then $(I - T_f)$ is demiclosed. We then apply Theorem 3. (For further applications of the theory of monotone operators to the study of nonexpansive mappings, see Browder [3], [5].) The same conclusion is obtained for Banach spaces X having a weakly continuous duality mapping J (e.g. the spaces l^p for $1 < p < +\infty$) in Browder [7] using the theory of J-monotone operators. q.e.d.

THEOREM 5. *Let X be a uniformly convex Banach space, T a nonexpansive self-mapping of X with a nonempty set F of fixed points. For a given constant λ with $0 < \lambda < 1$, let $S_\lambda = \lambda I + (1 - \lambda)T$.*

Then S_λ is asymptotically regular and has the same fixed points as T.

Hence the fixed points of T can be obtained from iteration of S_λ, *for which the conclusions of Theorems* 1–4 *can be applied.*

PROOF OF THEOREM 5. It is obvious that the fixed point sets of T and S_λ coincide and that S_λ is also a nonexpansive self-mapping of X.

Let u be a fixed point of T, and for a given x in X, let $x_n = S_\lambda^n x$. Since S_λ is nonexpansive and u is a fixed point of S_λ, it follows that $\|x_{n+1}-u\| \leqq \|x_n - u\|$ for all n, and hence that $\|x_n - u\|$ converges to a nonnegative limit d_0. Suppose that $d_0 > 0$. Since

$$x_{n+1} - u = S_\lambda(x_n) - u = \lambda(x_n - u) + (1 - \lambda)(Tx_n - u)$$

and since

$$\| x_n - u\| \to d_0, \qquad \|x_{n+1} - u\| \to d_0, \qquad \|Tx_n - u\| \leqq \|x_n - u\|,$$

it follows from the uniform convexity of X that

$$\|(x_n - u) - (Tx_n - u)\| \to 0,$$

i.e. $x_n - Tx_n \to 0$ strongly in X. Hence $x_{n+1} - x_n \to 0$ strongly in X, i.e. S_λ is asymptotically regular. q.e.d.

REMARK. For compact nonexpansive mappings T, the mapping S_λ with $\lambda = 1/2$ and its iterates were first studied by Krasnoselskiĭ [**10**]. For general λ, these mappings have been studied for compact T by Schaefer [**12**] and for demicompact T by Petryshyn [**11**]. All these results follow from the following:

THEOREM 6. *Let X be a Banach space, T a nonexpansive mapping of X into X which is asymptotically regular. Suppose that the fixed point set F of T is nonempty and that* $(I-T)$ *maps bounded closed subsets of X into closed subsets of X.*

Then for each x_0 *in X, the sequence* $T^n x_0$ *converges strongly in X to a fixed point of T.*

PROOF OF THEOREM 6. If u is a fixed point of T, $\|T^n x_0 - u\|$ does not increase with n. It suffices therefore to show that there exists a subsequence of $T^n x_0$ which converges strongly to a fixed point of T. Let G be the strong closure of the set $\{T^n x_0\}$. By the asymptotic regularity of T, $(I-T)(T^n x_0)$ converges strongly to 0 as $n \to +\infty$. Hence 0 lies in the strong closure of $(I-T)(G)$, and since the latter is closed by hypothesis since G is closed and bounded, 0 lies in $(I-T)(G)$. Hence there exists a strongly convergent subsequence of $\{T^n x_0\}$ which converges to an element v of G such that $(I-T)v = 0$, i.e. v is a fixed point of T. q.e.d.

REMARK. The hypothesis that $(I-T)$ maps bounded closed subsets of X into closed subsets of X is equivalent to the *demicompactness* of Petryshyn [11]. It is a consequence in particular of the stronger assumption that T is compact, i.e. that T maps bounded subsets of X into precompact subsets of X.

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