AN ASYMPTOTIC FORMULA FOR THE EIGENVALUES OF THE LAPLACIAN OPERATOR IN AN UNBOUNDED DOMAIN

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Communicated by F. Browder, February 10, 1966

F. Rellich [5] and, more generally, A. M. Molcanov [3] have shown that the problem

\[ \frac{1}{2} \Delta^2 u(x) + \lambda u(x) = 0, \quad x \in \Omega \]
\[ u(x) = 0, \quad x \in \partial \Omega \]

has a discrete spectrum (and consequently a complete orthonormal system of eigenfunctions in \( L^2(\Omega) \)) provided that \( \Omega \) is a "quasi-bounded" domain in \( E_n \). A domain \( \Omega \) is said to be quasi-bounded if

\[ \lim \text{dist}(x, \partial \Omega) = 0. \]

(See [1] for a proof of Molcanov's result, based on a generalization of the Kondrachoff embedding theorem for the Sobolev spaces \( H^m_0(\Omega) \).) The problem of determining the asymptotic behavior of the eigenvalues of (1) has remained open (cf. [2, p. 233]).

In the present note we consider the above problem from the point of view of random processes, as described in detail for the case of a bounded domain, as well as for the case of the operator \(-\frac{1}{2} \Delta^2 + V(x)\) (with \( V(x) \to +\infty \) as \( |x| \to \infty \)) on an unbounded domain, in the papers of D. Ray [4] and M. Rosenblatt [6]. We will show that if \( \Omega \) satisfies the following condition

\[ m(\Omega \cap [a < |x| < a + 1]) = O(a^{-\beta}) \]

for some \( \beta > \frac{1}{2} \), then simple modifications of Ray's arguments suffice to prove discreteness of the spectrum, as well as to obtain an asymptotic formula for the eigenvalues.

We take Ray's paper [4] as a starting point. Thus (assuming a cone condition for \( \Omega \), as described in Theorem 1 below) we already have a Green's function \( K(x, y, t) \) corresponding to the equation

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1 Research supported in part by the United States Air Force Office of Scientific Research, grant number AF-AFOSR-379-65.
and zero boundary conditions. We first wish to verify that the integral operator $K_t$ with kernel $K(x, y, t)$ is completely continuous in $L_2(\Omega)$. As in [4, Lemma 3] we see that it is sufficient to show, for fixed $t > 0$,

\begin{equation}
\int_{\Omega \cap \{|x| > a\}} |K_t(x)|^2 \, dx \to 0 \quad \text{as } a \to \infty,
\end{equation}

uniformly for $\psi \in L_2(\Omega), \|\psi\| = 1$. But, as in [4],

\begin{align*}
\int_{\Omega \cap \{|x| > a\}} |K_t(x)|^2 \, dx &\leq \int_{\Omega \cap \{|x| > a\}} dx \cdot \text{prob}\{x + x(t) \in \bar{\Omega}, 0 \leq \tau \leq t\} \cdot \|\psi\|^2.
\end{align*}

By an elementary calculation using (2), we have for any $\beta' < \beta$

\begin{align*}
\text{prob}\{x + x(t) \in \bar{\Omega}, 0 \leq \tau \leq t\} &\leq \text{prob}\{x + x(t) \in \bar{\Omega}\} = O(|x|^{-\beta'}), \quad x \in \Omega;
\end{align*}

here $t$ is fixed. Hence, writing $\Omega_i = \Omega \cap [i < |x| < i + 1], i = 0, 1, 2, \cdots$, we have (taking $\beta' > \frac{1}{2}$)

\begin{align*}
\int_{\Omega} \text{prob}\{x + x(t) \in \bar{\Omega}, 0 \leq \tau \leq t\} \, dx = \sum_i \int_{\Omega_i} \text{prob}\{x + x(t) \in \bar{\Omega}, 0 \leq \tau \leq t\} \, dx \\
= O\left(\sum_i i^{-2\beta'}\right) < \infty,
\end{align*}

and (4) follows from this. We therefore have

**Theorem 1.** Let $\Omega$ be an open set in $E_n$, satisfying condition (2) and the following cone condition: for each $x \in \partial \Omega$ there is an open cone with vertex $x$, lying outside $\bar{\Omega}$. Let $K_t$ be the integral operator in $L_2(\Omega)$ with kernel $K(x, y, t)$.

Then $K_t$ is completely continuous and hence has a countable set of eigenvalues \(\{\exp(-\lambda_j t), j = 0, 1, 2, \cdots\}\) with corresponding complete orthonormal eigenfunctions \(\{\phi_j(x)\}\), which are independent of $t$. Moreover the $\lambda_j$ are eigenvalues and the $\phi_j$ eigenfunctions of the problem (1).
Corollary. Let $\Omega$ be as in Theorem 1. Then

$$\sum_{\lambda_j < \lambda} \phi_j^2(x) \sim \left( \frac{\lambda}{2\pi} \right)^{n/2} \frac{1}{\Gamma(1 + n/2)}$$

as $\lambda \to \infty$, for each $x \in \Omega$.

The proofs of the asserted properties of the $\lambda_j$ and $\phi_j$ are the same as in Ray's paper. In particular, Ray shows that

$$\sum_j \exp(-\lambda_j t)\phi_j^2(x) = K(x, x, t) \sim \left( \frac{1}{2\pi t} \right)^{n/2}$$

as $t \to 0$, uniformly for $x \in \Omega$; in the present case this follows from the fact that $K(x, y, t)$ is a Hilbert-Schmidt kernel, as can be proved in a manner similar to the above verification of (4)—it is useful to notice that $0 \leq K(x, y, t) \leq (2\pi t)^{-n/2} \exp(-|x-y|^2/2t)$.

Theorem 2. Let $\Omega \subset E_n$ satisfy the hypotheses of Theorem 1. Let $\rho(x)$ be a nonnegative function in $L_1(\Omega)$. Define

$$N_\rho(\lambda) = \sum_{\lambda_j \leq \lambda} \int_\Omega \rho(x)\phi_j^2(x) \, dx.$$

Then

$$N_\rho(\lambda) \sim \left( \frac{\lambda}{2\pi} \right)^{n/2} \frac{1}{\Gamma(1 + n/2)} \int_\Omega \rho(x) \, dx.$$

Proof. Applying (5) to the Laplace-Stieltjes transform of $N_\rho(\lambda)$, we have

$$\int_0^\infty e^{-\lambda t} \, dN_\rho(\lambda) = \int_\Omega \rho(x) \sum_j \exp(-\lambda_j t)\phi_j^2(x) \, dx$$

$$= \int_\Omega \rho(x) K(x, x, t) \, dx$$

$$\sim \left( \frac{1}{2\pi t} \right)^{n/2} \int_\Omega \rho(x) \, dx.$$

Hence the Tauberian theorem of Karamata applies, and yields (6).

q.e.d.

We obviously obtain the classical formula of Weyl if we put $\rho(x) \equiv 1$ on a bounded region (or even on an unbounded region of finite volume). If in the general case we choose a bounded function $\rho(x)$, we obtain $N_\rho(\lambda) \leq c \cdot N(\lambda)$ where $N(\lambda) = N_1(\lambda)$ is the usual function; we
therefore obtain a one-sided estimate for $N(\lambda)$:

$$N(\lambda) \geq \lambda^{n/2}$$

where $f(\lambda) \geq g(\lambda)$ means the same as $g(\lambda) = O(f(\lambda))$. We remark that our results are unaffected if the operator $-\Delta^2$ is replaced by $-\Delta^2 + V(x)$ if $V(x)$ is a bounded function on $\Omega$.

The foregoing results can also be derived using analytical methods similar to those of Titchmarsh [7]; the basic properties of the Green’s function $G(x, y, \lambda)$ in this case are due to D. Hewgill (Thesis, University of British Columbia).

REFERENCES

5. F. Rellich, *Das Eigenwertproblem von $\Delta u + \lambda u = 0$ in Halbröhren*, in Essays presented to R. Courant, Interscience, New York, 1948, 329–344.

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