ON POINCARÉ’S BOUNDS FOR HIGHER EIGENVALUES

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Communicated by A. Zygmund, February 23, 1966

1. Introduction. Let \( A \) be a compact symmetric negative-definite operator on a real Hilbert space \( H \) having the inner product \((u, v)\). Let \( \lambda_1 \leq \lambda_2 \leq \cdots \) be the eigenvalues and \( u_1, u_2, \cdots \) the corresponding orthonormal set of eigenvectors of the equation \( Au = \lambda u \). Denote by \( R(u) \) the Rayleigh quotient \((Au, u)/(u, u)\).

For a given \( \lambda_n \) let \( m \) and \( N \) be the smallest and largest indices respectively such that \( \lambda_m = \lambda = \lambda_N \). There are two variational characterizations of \( \lambda_n \) by inequalities. One goes back to Poincaré [1, p. 259] and was reformulated by Pólya and Schiffer [2], [3]. The other is the maximum-minimum principle for which A. Weinstein [4], [5] recently introduced a new approach. Using the Weinstein determinant and the corresponding quadratic form he gave for the first time a complete discussion of the corresponding inequalities including the necessary and sufficient conditions for equality. In the present paper we give a similar discussion of Poincaré’s characterization of \( \lambda_n \).

2. The main result. Let \( V_r \) be any \( r \)-dimensional subspace of \( H \) and let \( p_1, p_2, \cdots, p_r \) be a basis for \( V_r \). We consider the determinant

\[
\det \{ (A p_i, p_k) - \lambda (p_i, p_k) \}, \quad i, k = 1, 2, \cdots, r.
\]

Using Parseval’s formula we see that (1) can also be written as

\[
\det \left\{ \sum_{j=1}^{\infty} (\lambda_j - \lambda) (p_i, u_j) (p_k, u_j) \right\}, \quad i, k = 1, 2, \cdots, r.
\]

Let us note in passing the remarkable, but until now unexplained, similarity between (2) and the Weinstein determinant

\[
W(\lambda) = \det \left\{ \sum_{j=1}^{\infty} (\lambda_j - \lambda)^{-1} (p_i, u_j) (p_k, u_j) \right\}, \quad i, k = 1, 2, \cdots, r.
\]

We can now formulate our main result.

**Theorem.** For any choice of \( V_r \) we have the inequality

\[1\] This paper was prepared by the author, while the author was an NDEA fellow in the Institute for Fluid Dynamics and Applied Mathematics of the University of Maryland.

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\( \lambda_n \leq \max_{u \in V_r} R(u) \)

if and only if \( m \leq r \). By varying \( V_r \) we obtain the following characterization of \( \lambda_n \).

\[ \lambda_n = \min_{V_r} \max_{u \in V_r} R(u), \quad m \leq r \leq N. \]

Assuming that \( m \leq r \), the necessary and sufficient conditions on the space \( V_r \) for the equality

\[ \lambda_n = \max_{u \in V_r} R(u) \]

are that \( r \leq N \) and for any \( \epsilon > 0 \) the quadratic form with the symmetric matrix

\[ \{(A p_i, p_k) - (\lambda_n + \epsilon)(p_i, p_k)\}, \quad i, k = 1, 2, \ldots, r \]

is negative definite.

**Proof.** The proofs of (4) and (5) have been given in [1] and [2], [3] for the case \( r = n \). Obviously (4) holds also for \( m \leq r \) since \( \lambda_m = \lambda_n \). To show the necessity of this condition we assume for the moment that (4) holds for all \( V_r \) where \( r < m \) and choose \( V_r \) to be the subspace spanned by \( u_1, u_2, \ldots, u_r \). In this case we have

\[ \max_{u \in V_r} R(u) = \lambda_r < \lambda_m = \lambda_n \leq \max_{u \in V_r} R(u) \]

which is a contradiction. As in [2], [3] the equality (5) follows immediately not only for \( r = n \) but also for \( m \leq r \leq N \). In fact, it is sufficient to use the classical choice \( p_k = u_k \), \( k = 1, 2, \ldots, r \) in order to obtain (6). In §3 we give an example which shows that the classical choice is not a necessary condition for (6). To prove our necessary and sufficient conditions we shall assume that the basis \( p_1, p_2, \ldots, p_r \) has been chosen so that the matrix (7) is diagonal. First we show that our conditions are necessary. Suppose that (6) holds for \( r > N \). Then, using (4), we obtain the contradiction

\[ \lambda_r \leq \max_{u \in V_r} R(u) = \lambda_n = \lambda_N < \lambda_r. \]

Since (6) implies

\[ R(p_i) = (A p_i, p_i)/(p_i, p_i) < \lambda_n + \epsilon, \quad i = 1, 2, \ldots, r \]

all elements on the diagonal of (7) are negative, which proves that the quadratic form corresponding to (7) must be negative definite. To
prove sufficiency we assume that for any $\epsilon > 0$ the diagonal matrix (7) is negative definite so that

$$ (A\phi_i, \phi_i) < (\lambda_n + \epsilon)(\phi_i, \phi_i), \quad i = 1, 2, \ldots, r $$

and

$$ (A\phi_i, \phi_k) = (\lambda_n + \epsilon)(\phi_i, \phi_k), \quad i \neq k; \quad i, k = 1, 2, \ldots, r. $$

Since every $u \in V$, can be written as $u = \sum_{i=1}^{r} \gamma_i \phi_i$ we have

$$ R(u) = \frac{\sum_{i=1}^{r} \gamma_i^2 (A\phi_i, \phi_i) + \sum_{i \neq k} \gamma_i \gamma_k (A\phi_i, \phi_k)}{\sum_{i,k=1}^{r} \gamma_i \gamma_k (\phi_i, \phi_k)}. $$

Using (8) and (9) in (10) we get for every $u \in V$, $R(u) < \lambda_n + \epsilon$. Combining this with (4) we have $\lambda_n \leq \max_{u \in V} R(u) \leq \lambda_n + \epsilon$. Since $\epsilon$ can be chosen arbitrarily small the equality (6) holds.

3. **Example.** We now give an example in which (6) holds for a non-classical choice of $V_r$. Let $\lambda_1 < \lambda_2 < \lambda_3$ and let $m = r = n = N = 2$. We choose $p_1 = u_2$ and $p_2 = u_1 + \beta u_3$ as a basis for $V_2$ where $0 < \beta^2 \leq (\lambda_2 - \lambda_1)/(\lambda_3 - \lambda_2)$. A simple calculation shows that for every $u \in V_2$ the inequality $R(u) \leq \lambda_2$ is satisfied. Since $R(u_2) = \lambda_2$ we have $\lambda_2 = \max_{u \in V_2} R(u)$. In this case (7) is a diagonal matrix with elements $-\epsilon, -\epsilon(1 + \beta^2)$, which verifies our criterion. Let us note the formal analogy to the new maximum-minimum theory of A. Weinstein, where the quantities $(\lambda_j - \lambda)^{-1}$, $\lambda_n - \epsilon$, and $\beta^{-1}$ appear in place of $\lambda_j - \lambda$, $\lambda_n + \epsilon$, and $\beta$.

4. **Concluding remark.** It has been shown in [1] and [2], [3] that the roots $\lambda'_1 \leq \lambda'_2 \leq \cdots \leq \lambda'_r$ of (1) satisfy the inequalities

$$ \lambda_1 \leq \lambda'_1, \quad \lambda_2 \leq \lambda'_2, \quad \lambda_r \leq \lambda'_r $$

and that the simultaneous equalities

$$ \lambda_1 = \lambda'_1, \quad \lambda_2 = \lambda'_2, \quad \lambda_r = \lambda'_r $$

are obtained by choosing $p_k = u_k$, $k = 1, 2, \ldots, r$. In another paper we shall prove that the only $V_r$ for which (11) holds are those subspaces generated by eigenvectors belonging to $\lambda_1, \lambda_2, \ldots, \lambda_r$.

**References**


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