ON THE MAXIMAL RING OF QUOTIENTS OF \( C(X) \)

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1. Let \( Q(X) \) denote the maximal ring of quotients (in the sense of Johnson \([4]\) and Utumi \([5]\)) of the ring \( C(X) \) of continuous real-valued functions on the completely regular Hausdorff space \( X \). This ring has been studied by Fine, Gillman, and Lambek \([1]\) and realized by them as the direct limit of the subrings \( C(V) \), \( V \) a dense open subset of \( X \) (i.e., the union of these \( C(V) \)'s, modulo the obvious equivalence relation). From this representation of \( Q(X) \), it follows that if \( X \) and \( Y \) have homeomorphic dense open subsets, then \( Q(X) \) and \( Q(Y) \) are isomorphic. The full converse to this is false (see below). In this note a proof of the following is described.

**Theorem 1.** Let \( X \) and \( Y \) be separable metric spaces. If \( Q(X) \) and \( Q(Y) \) are isomorphic, then \( X \) and \( Y \) have homeomorphic dense open subsets.

In particular, the spaces \( \mathbb{R}^n \), \( n = 1, 2, \ldots \) (\( \mathbb{R} = \) the reals) have pairwise nonisomorphic \( Q \)'s, thus settling a question\(^1\) raised in \([1]\). That \( Q(\mathbb{R}) \) is not isomorphic to \( Q(\mathbb{R}^n) \), for \( n > 1 \), was shown by F. Rothberger and J. Fortin. (See \([2]\), and \([1, \text{p. } 16]\).)

The main purpose of this note is to present a fairly simple solution to this question, and therefore the possible generalizations of Theorem 1 will not be discussed here. These generalizations, and related questions, will be treated in detail in a later paper.

The proof of Theorem 1 will now be described.

2. Homomorphisms of \( C(Y) \) into \( C(X) \) are well understood \([3, \text{Chapter 10}]\). If \( \tau: X \to Y \) is continuous, \( \phi(f) = f \circ \tau \) defines a homomorphism \( \phi: C(Y) \to C(X) \). Conversely, if \( Y \) is realcompact, and \( \phi: C(Y) \to C(X) \) is a homomorphism with \( \phi(1) = 1 \), then \( \phi \) is induced by a continuous function in this manner.

Now, let \( W_0 \) be a dense open subset of \( X \), and let \( \tau: W_0 \to Y \) be continuous and additionally satisfy: for each dense open subset \( V \) of \( Y \), \( \tau^{-1}[V] \) is dense in \( X \). Then \( \phi(f) = f \circ \tau \) defines a homomorphism \( \phi: Q(Y) \to Q(X) \). Evidently, \( \phi \) satisfies

\[ (*) \text{ for each dense open subset } V \text{ of } Y, \text{ there is a dense open subset } W \text{ of } X \text{ such that } \phi[C(V)] \subset C(W). \]

\(^1\) The author is indebted to Professor Nathan J. Fine for communicating this question, and for many valuable conversations concerning it.
Conversely, if $Y$ is hereditarily realcompact, and $\phi: Q(Y) \to Q(X)$ is a homomorphism with $\phi(1) = 1$, and satisfying (*) then $\phi$ is induced by a continuous function in the manner described.

(*) states that $\phi$ respects the direct limit representations for the $Q$'s. A homomorphism satisfying (*) will be called a dl-homomorphism, and an isomorphism $\phi$ such that both $\phi$ and $\phi^{-1}$ satisfy (*), a bi-dl-isomorphism.

**Proposition 2.** Let $X$ and $Y$ be hereditarily realcompact. $Q(X)$ and $Q(Y)$ are isomorphic by a bi-dl-isomorphism iff $X$ and $Y$ have homeomorphic dense open subsets.

The situation with $X$ and $\beta X$ is interesting. For $V$ a dense open subset of $\beta X$, and $f \in C(V)$, define $\phi(f) = f | V \cap X$. A dl-isomorphism $\phi: Q(\beta X) \to Q(X)$ results. In [1] it is shown that each continuous function on a dense open subset of $X$ is extendible to a continuous function on a dense open supset of $\beta X$; hence $\phi$ is onto $Q(X)$. $\phi^{-1}$ is a dl-isomorphism iff each dense open subset of $X$ is $C$-embedded in some dense open subset of $\beta X$. Choose for $X$ the rationals $P$. Like any realcompact space, $P$ is $C$-embedded in no space in which $P$ is dense, and $\phi^{-1}$ is not a dl-isomorphism. In fact, there is no dl-isomorphism of $Q(P)$ onto $Q(\beta P)$, for it can be shown that such a mapping would be induced by a homeomorphism of a dense open subset of $\beta P$ onto a subset of $P$, and such homeomorphisms do not exist.

3. For $f, g \in Q(X)$, define $f \geq g$ if $f(x) \geq g(x)$ for all $x \in \text{dom } f \cap \text{dom } g$. $Q(X)$ is thus a partially ordered ring. $Q^*(X)$ (the subring of bounded functions) is a metric space under

$$
\rho(f, g) = \sup \{ |f(x) - g(x)| : x \in \text{dom } f \cap \text{dom } g \}.
$$

(For much more on these matters, see [1].)

**Lemma 3.** Let $X$ be separable and first-countable. Let $A$ be a lattice subring of $Q(X)$ which contains constants and is closed under bounded inversion (i.e., if $f \in A$ and $f \geq 1$, then $1/f \in A$); let $A^*$ be complete (metrically, under $\rho$). Then there is a dense open subset $V$ of $X$ with $A \subset C(V)$.

The proof of this goes as follows. If, for every $V$, $A \subset C(V)$ then the “singularities” of the functions in $A$ are dense in some open set $G$; hence, some countable subset $\{p_1, p_2, \ldots \}$ of these singularities is also dense in $G$. For each $n, f_n \in A^*$ can be found for which the oscillation of $f_n$ at $p_n$ is nonzero. Upon suitable choice of real numbers $a_1, a_2, \ldots$, the partial sums of $\sum_{n=1}^{\infty} a_nf_n$ form a Cauchy sequence in $A^*$ with no limit in $Q(X)$. 

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Theorem 1 is easily proved from Lemma 3. Let \( X \) and \( Y \) be separable metric spaces, and let \( \phi \) be an isomorphism of \( Q(Y) \) onto \( Q(X) \). By routine arguments (\( \phi \) preserves order, etc.), for any dense open subset \( V \) of \( Y \), \( \phi[C(V)] \) satisfies the hypotheses of Lemma 3. Hence \( \phi \) (and by the same reasoning, \( \phi^{-1} \)) is a dl-isomorphism. Proposition 2 now applies.

REFERENCES