MULTIPLICATION IN GROTHENDIECK RINGS
OF INTEGRAL GROUP RINGS

BY D. L. STANCL

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1. Introduction. Let $G$ be a finite group, $Z$ the ring of rational integers, and form the Grothendieck ring $K^0(ZG)$ of the integral group ring $ZG$. Swan [4] has described multiplication in $K^0(ZG)$ when $G$ is cyclic of prime power order. The purpose of this note is to present results which describe multiplication in $K^0(ZG)$ when $G$ is cyclic or elementary abelian. Full details will appear elsewhere.

Let $Q$ denote the rational field, and recall that the elements of $K^0(QG)$ are $Z$-linear combinations of symbols $[M^*]$, where $M^*$ ranges over all finitely-generated left $QG$-modules, and similarly for $K^0(ZG)$. We define a ring epimorphism $\theta: K^0(ZG) \to K^0(QG)$ by $\theta[M] = [Q \otimes_Z M]$, and call any linear mapping $f: K^0(QG) \to K^0(ZG)$ such that $\theta f = 1$ a lifting map for $K^0(ZG)$. Since the Jordan-Hölder Theorem holds for $QG$-modules, $K^0(QG)$ is the free abelian group with basis $\{[M^*_i]: 1 \leq i \leq m\}$, where $\{M^*_i: 1 \leq i \leq m\}$ is a full set of non-isomorphic irreducible $QG$-modules. Swan [4] has shown that to describe multiplication in $K^0(ZG)$ it suffices to describe the products $f[M^*_i] \cdot f[M^*_j]$ for $1 \leq i, j \leq m$, and $f[M^*_i]x$, for $1 \leq i \leq m$ and $x \in \ker \theta$.

2. Statement of results. Let $G$ be cyclic of order $n$ with generator $g$. For each $s$ dividing $n$, $\zeta_s$ will denote a primitive $s$th root of unity, and $Z_s$ will denote the $ZG$-module $Z[\zeta_s]$ on which $g$ acts as $\zeta_s$. Similarly, $Q_s$ will denote the $QG$-module $Q(\zeta_s)$. Then $K^0(QG)$ is the free abelian group with basis $\{[Q_s]: s \divides n\}$, and $f: K^0(QG) \to K^0(ZG)$ by $f[Q_s] = [Z_s]$ is a lifting map. Swan [4] has shown that $f$ is a ring homomorphism. Also, for each $s$ dividing $n$, $G_s$ will denote the quotient group of $G$ of order $s$, and if $s \divides t$, $N_{s/t}$ will denote the norm from $Q_s$ to $Q_t$. By the results of Heller and Reiner [2],

$$\ker \theta = \left\{ \sum_{s \divides n} ([A_s] - [Z_s]): A_s = Z_s\text{-ideal in } Q_s \right\}.$$

**Theorem 1.** Multiplication in $K^0(ZG)$ is given by the formula

$$[ZG_r][A_s] = \sum_d ([N_{s/r}(A_s)Z_d] - [Z_d]),$$

for all $r, s$ dividing $n$, where $s' = s/(r, s)$ and $d$ ranges over all divisors of $[r, s]$ such that $(r, s)/d, s' = 1.$
MULTIPLICATION IN GROTHENDIECK RINGS

THEOREM 2. If $G$ is an elementary abelian group, multiplication in $K^0(ZG)$ can be explicitly determined.

We remark that it is possible to give formulas which describe multiplication in $K^0(ZG)$ when $G$ is elementary abelian. These formulas will not be included here.

3. Proof of Theorem 1. We first suppose that $r = p^a$, for some prime $p$ and nonnegative integer $a$, and write $s = p^b$, $(p, t) = 1$. If $a = 0$ or $b = 0$, the theorem is trivial. Let $Z = Z_s/A_*$ and for each $i$ dividing $s$, let $Z_i$ denote the $Z$-module $Z$ on which $g$ acts as $x_i$ reduced modulo $A_*$. It suffices to find $M = ZG_{x} \otimes Z \hat{Z}$. Since $ZG_{x} \cong Z[x]/(x^a - 1)$, $M \cong \hat{Z}(x)/(x^a - 1)$. If $a \leq b$, then in $\hat{Z}[x]$, $x^a - 1 = \prod_k (x - x_i^b)$, $1 \leq k \leq p^a$, and thus $M \cong \sum_k \hat{Z}(x_i^b)$. A calculation with norms now yields the desired result. If $a > b$, then $x^a - 1$ factors in $\hat{Z}[x]$ as follows: $x^a - 1 = \prod_k (x - x_i^b) \prod_{i,j} (x^{p^i} - x_j^b)$, where $1 \leq k \leq p^a$, $b + 1 \leq i \leq a$, and $1 \leq j \leq p^b$ with $(p, j) = 1$. Therefore

$$M \cong \sum_k \hat{Z}(x_i^{p_k}) + \sum_{i,j} (Z_{x_i^{p_k}}/A_*Z_{x_i^{p_k}})(x_j^{p_k})$$

where $(Z_{x_i^{p_k}}/A_*Z_{x_i^{p_k}})(x_j^{p_k})$ denotes the $Z$-module $Z_{x_i^{p_k}}/A_*Z_{x_i^{p_k}}$ on which $g$ acts as $x_j^{p_k}$. Again, a calculation with norms will yield the desired result. This proves the theorem for the case $r = p^a$. The general case follows by the use of induction on the number of distinct prime divisors of $r$.

4. Proof of Theorem 2. In order to prove Theorem 2, we need several lemmas.

LEMMA 1. Let $G$ be an abelian group, $F$ an algebraic number field which is a splitting field for $G$, and $R$ the ring of algebraic integers of $F$. Then multiplication in $K^0(RG)$ can be explicitly determined.

Let $G$ be an elementary abelian group and write $G = G_1 \times \cdots \times G_k$, where $G_i$ is cyclic of order $p$ with generator $g_i$, for $1 \leq i \leq k$. Let $\xi$ be a primitive $p$th root of unity, $F = Q(\xi)$, $R = Z[\xi]$, and denote by $F(a_1, \cdots, a_k)$ the FG-module $F$ on which $g_i$ acts as $\xi^a_i$, where $1 \leq a_i \leq p$ for $1 \leq i \leq k$. Similarly, if $A$ is any $R$-ideal in $F$, $A(a_1, \cdots, a_k)$ will denote the $R$-module $A$ on which $g_i$ acts as $\xi^a_i$. Note that, by restriction of operators, $F(a_1, \cdots, a_k)$ and $A(a_1, \cdots, a_k)$ are QG- and Z-modules, respectively. It is easy to prove that the QG-modules of form $F(\phi, \cdots, \phi, p, 1, a_{j+1}, \cdots, a_k)$, where $1 \leq j \leq k$, together with the trivial module $Q$, form a full set of nonisomorphic irreducible QG-modules.

Define $\psi: K^0(ZG) \rightarrow K^0(RG)$ by $\psi[Y] = [R \otimes Z Y]$, where $R \otimes Z Y$ is an RG-module with action of $R$ given by $r'(r \otimes y) = r'r \otimes y$, for all
r' \in R$, and action of $G$ given by $g(r \otimes y) = r \otimes gy$, for all $g \in G$. Similarly, define $\eta: K^0(QG) \to K^0(FG)$ by $\eta[y^*] = [F \otimes_Q Y^*]$.

**Lemma 2.** $\psi$ and $\eta$ are ring homomorphisms and the following diagram commutes and is exact:

\[
0 \longrightarrow \ker \theta_R \longrightarrow K^0(RG) \overset{\theta_R}{\longrightarrow} K^0(FG) \longrightarrow 0
\]

\[
0 \longrightarrow \ker \theta_Z \longrightarrow K^0(ZG) \overset{\theta_Z}{\longrightarrow} K^0(QG) \longrightarrow 0
\]

Let $\Phi_p(x)$ denote the cyclotomic polynomial of order $p$. If we apply $\psi$ to $[A(p, \ldots, p, 1, a_{j+1}, \ldots, a_k)] \subseteq K^0(ZG)$, we note that $\Phi_p(g_j)$ annihilates $R \otimes_z A(p, \ldots, p, 1, a_{j+1}, \ldots, a_k)$. Since $\Phi_p(x)$ splits into linear factors in $R[x]$, this gives us a composition series for $R \otimes_z A(p, \ldots, p, 1, a_{j+1}, \ldots, a_k)$. If we denote by $A'^{(t)}$ the ideal conjugate to $A$ under the $Q$-automorphism of $F$ which takes $\xi$ into $\xi^t$, we thus obtain

\[
\psi[A(p, \ldots, p, 1, a_{j+1}, \ldots, a_k)] = \sum_t [A'^{(t)}(p, \ldots, p, t, a_{j+1}, \ldots, a_k)], \text{ where } 1 \leq t \leq p - 1.
\]

We now use the formulas for $\ker \theta_Z$ and $\ker \theta_R$ obtained by Heller and Reiner [2], and our formula for $\psi[A(p, \ldots, p, 1, a_{j+1}, \ldots, a_k)]$, to show that $\psi: \ker \theta_Z \to \ker \theta_R$ is monic. Lemma 2 then implies that $\psi: K^0(ZG) \to K^0(RG)$ is monic. Now define $f_R: K^0(FG) \to K^0(RG)$ by $f_R[F(a_1, \ldots, a_k)] = [R(a_1, \ldots, a_k)]$. It is clear that $f_R$ is a lifting map for $K^0(RG)$, and it is easy to show that $f_R$ is a ring homomorphism. Since $\psi$ is monic, we may define $f_Z = \psi^{-1}f_R \eta$. Then $f_Z$ is a lifting map for $K^0(ZG)$ and is a ring homomorphism. Finally, since $F$ is a splitting field for $G$, multiplication in $K^0(RG)$ is known by Lemma 1, and hence multiplication in $K^0(ZG)$ can be explicitly determined by the use of the monomorphism $\psi$. This completes the proof of Theorem 2.

**Bibliography**


**University of Illinois**