COHOMOLOGY OF ALGEBRAIC GROUPS AND INVARIANT SPLITTING OF ALGEBRAS\(^1, 2\)

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1. Introduction. Let $A$ be an algebra, over a field $F$, assumed at first to be associative and finite-dimensional over $F$. Let $R$ be the radical of $A$, $C$ the center of $A$. Assume $A/R$ separable, so that $A$ possesses maximal separable subalgebras (Wedderburn factors) $S$ for which $A = S + R$, $S \cap R = 0$. Let $G$ be a group of automorphisms and antiautomorphisms of $A$. We will discuss the existence and uniqueness of $G$-invariant Wedderburn factors in terms of various cohomology groups of $G$. In general, the cohomology is that of abstract groups. However, the conditions given will be compatible with taking the algebraic hull of $G$ (in the Zariski topology with respect to $F$), so that we can assume $G$ is an algebraic group and the cohomology is rational. We will outline here how the cohomology enters. Details will appear elsewhere. See [3], [4], [5] for a general background of the question.

2. Existence. We first assume $R^2 = 0$. Let $S$ be any maximal separable subalgebra. If $g \in G$, then $Sg$ is another maximal separable subalgebra, so by the Malcev theorem, $Sg = SC_{g^{-1}}$, where $C_w$ is conjugation by $w$. $z(g)$ is in $R$, but is uniquely determined modulo $R \cap C$, so that we consider $z$ as a function from $G$ to the vector space $R/R \cap C$. We consider $R/R \cap C$ as a $G$-module in the obvious way, except that the antiautomorphisms in $G$ act via their negatives. Then a technical calculation will show that $z \in Z^1(G, R/R \cap C)$, i.e., $z(gh) = z(g) \cdot h + z(h)$. Hence if $H^1(G, R/R \cap C) = 0$, there is an $x$ in $R$ such that $z(g) = x - x \cdot g + R \cap C$. A technical calculation will then show that $SC_{1-x}$ is a $G$-invariant maximal separable subalgebra.

Now we consider the general case $R^2 \neq 0$. The action of $G$ on all modules will be the obvious ones, except that the antiautomorphisms in $G$ will act via their negatives. We consider $A/R^2$. The condition for the case $R^2 = 0$ above now becomes $H^1(G, R/\{x \in R \mid \{a, x\} \subseteq R^2\}) = 0$ where $\{a, x\} = \{ax - xa \mid a \in A\}$. If this holds, then $A = S_1 + R$, $S_1$ a $G$-invariant subalgebra, $S_1 \cap R \subseteq R^2$. $S_1$ has radical $R^2$, and we next consider $S_1/R^2$. The condition now is

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$H^1(G, R^2/\{ x \in R^2 \mid [S_i, x] \subseteq R^4 \}) = 0$. This yields $A = S_2 + R$, $S_2$ a $G$-invariant subalgebra, $S_i \cap R \subseteq R^4$. Let $R^{2n+1} \neq 0$, $R^{2n+2} = 0$. Then the conditions become $H^1(G, R^2/\{ x \in R^2 \mid [x, S_i] \subseteq R^{2n+1} \})$ for $i = 0, 1, \cdots, n$, where $S_0 = A$, $S_1, \cdots, S_n$ are $G$-invariant subalgebras as indicated. The next step yields $S_{n+1}$ as a $G$-invariant maximal separable subalgebra.

3. **Applications.** All the modules considered are rational modules for the algebraic hull of $G$, and the cocycles are rational functions. Hence we may assume $G$ is an algebraic group. If $G$ is reductive, then the rational cohomology $H^1(G, M) = 0$ for $M$ a rational $G$-module.

This follows from an argument in [1] as follows: Let $W = F \oplus M$, $f \in Z^1(G, M)$. Let $G$ act on $W$ by $(a, m)g = (a, mg + f(g))$. $W$ is completely reducible since $G$ acts rationally on it. Let $C$ be a $G$-complement to $M$ in $W$. $C$ has a unique element $(1, x)$, $x \in M$. Applying $g \in G$ yields $f(g) = x - xg$. This argument shows that $A$ possesses $G$-invariant maximal separable subalgebras if the algebraic hull of $G$ is a reductive algebraic group. In particular, it holds if $F$ has characteristic zero and $G$ is completely reducible (see [2]).

The cohomology conditions are well-known if $G$ is a finite group of order not divisible by the characteristic of $F$.

Note that $\{ x \in R^2 \mid [x, S_i] \subseteq R^{2n+1} \}$ is a Lie ideal in $S_i$. This indicates that similar results hold for Lie algebras over fields of characteristic zero.

By inducting on the degree of nilpotency of $R$, rather than on the dimension of $A$, we note that the cohomology conditions (for abstract groups) will suffice for infinite-dimensional algebras (with nilpotent radicals), provided the algebras involved possess Wedderburn principal decompositions which satisfy the Malcev theorem.

See also [5] for additional results concerning existence.

4. **Uniqueness.** Here the aim is to prove that any two $G$-invariant maximal separable subalgebras $S$ and $T$ are $G$-orthogonally conjugate (see [4] for definitions). Again, $G$ will act naturally on the modules which arise, except that the antiautomorphisms act via their negatives. Also let characteristic $F \neq 2$.

First let $R^2 = 0$, $z$ in $R$ so that $SC_{1-z} = T$. A technical calculation yields $z - z \cdot g$ in $R \cap C$, and $f(g) = z - z \cdot g$ satisfies $f$ in $Z^1(G, R \cap C)$. If $H^1(G, R \cap C) = 0$, then there is an $x$ in $R \cap C$ so $z - z \cdot g = x - x \cdot g$. Then $z - x$ is $G$-skew (called $G$-symmetric in [4]) so $1 - z + x$ is $G$-orthogonal, and $SC_{1-z^2} = T$.

Now consider the general case. The cohomology condition for $A/R^2$ becomes $H^1(G, \{ x \in R \mid [T, x] \subseteq R^2 \})/R^2 = 0$. Then this yields
$z_1$ in $R$ so that $z_1 + R^2$ is $G$-skew in $R/R^2$ and $SC_{1-z_1} + R^2 = T + R^2$.

Define $f_1(g) = \frac{1}{2}(z_1 \cdot g - z_1)$. Then $f_1 \in Z^1(G, R^2)$. If $H^1(G, R^2) = 0$, we get an $x_1$ in $R^2$ so $f_1(g) = x_1 - x_1 \cdot g$. Let $y_1 = -x_1 - z_1/2$. Then $y_1$ is $G$-skew and $y_1 + R^2 = -z_1/2 + R^2$. Let $u_1 = -2y_1(1 - y_i)^{-1}$. Then $1 - u_1$ is $G$-orthogonal and $SC_{1-u_1} + R^2 = T + R^2$.

Repeating these arguments, two sets of conditions emerge. They are (1) $H^i(G, \{x \in R^{2i} \mid [T, x] \subseteq R^{2i+1}\}/R^{2i+1}) = 0$, $i = 0, 1, \ldots, n$, and (2) $H^i(G, R^{2i}) = 0$, $i = 1, \ldots, n$. The first set yields $G$-skew cosets. The second set enables one to lift out suitable $G$-orthogonal elements out of which are built $G$-orthogonal elements using a Cayley transform. Finally $(1-u_1)(1-u_2) \cdot \cdot \cdot (1-u_n)(1-u_{n+1})$ conjugates $S$ into $T$.

The same general remarks in §3 apply here also. However, we point out that for finite-dimensional algebras the conclusion can be proved for any completely reducible group, even if the characteristic is not zero (but not two), as in [4].

References

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