FIXED POINT FREE INVOLUTIONS ON HOMOTOPY SPHERES

BY W. BROWDER AND G. R. LIVESAY[1]

Communicated by J. Milnor, November 9, 1966

1. Introduction and statements of theorems. Let $T: \Sigma^{n+1} \rightarrow \Sigma^{n+1}$ be a smooth[2] (C^∞) fixed point free involution on a smooth manifold, Σ^{n+1}, homeomorphic to the $(n+1)$-sphere, S^{n+1}. We wish to consider the following problem: does there exist an n-sphere, S^n, smoothly imbedded in Σ^{n+1} such that $TS^n = S^n$? If such an S^n exists, we will say that (T, Σ^{n+1}) *desuspends* to $(T|S^n, S^n)$ and that $(T|S^n, S^n)$ *suspends* to (T, Σ^{n+1}). We claim (proofs are to appear later):

THEOREM 1. *If $n \geqq 5$ is odd, then (T, Σ^{n+1}) desuspends to $(T|S^n, S^n)$ for some T-invariant $S^n \subset \Sigma^{n+1}$.*

If n is even, there are obstructions to desuspending (T, Σ^{n+1}). There is a bilinear form, $B(x, y)$ defined on a certain subgroup of $H_*(M)$, where $\Sigma^{n+1} = A \cup TA$, A and TA are compact submanifolds of Σ^{n+1} with smooth boundary, and $\partial A = \partial TA = A \cap TA = M$. If $n \equiv 2 \pmod 4$, then B is symmetric, and its signature, $\sigma(T, \Sigma^{n+1})$ is determined by (T, Σ^{n+1}). If $n \equiv 0 \pmod 4$, then B is skew-symmetric. Furthermore, if $n = 4k$, there is a map $\psi_0: H_{2k}(M; Z_2) \rightarrow Z_2$ such that $\psi_0(x+y) = \psi_0(x) + \psi_0(y) + B_2(x, y)$, where B_2, defined on a subgroup of $H_{2k}(M; Z_2)$, corresponds to B, defined on a subgroup of $H_{2k}(M)$. The Arf invariant, $c(T, \Sigma^{n+1})$, [**1**], [**4**], corresponding to ψ_0 and B_2, depends only on (T, Σ^{n+1}). Regarding these invariants, we have

THEOREM 2. *If $n \equiv 2 \pmod 4$ and $n > 5$, then (T, Σ^{n+1}) can be desuspended to $(T|S^n, S^n)$ if and only if $\sigma(T, \Sigma^{n+1}) = 0$.*

THEOREM 3. *If $n \equiv 0 \pmod 4$ and $n > 4$, then (T, Σ^{n+1}) can be desuspended to $(T|S^n, S^n)$ if and only if $c(T, \Sigma^{n+1}) = 0$.*

At present, we have no example of (T, Σ^{n+1}) for which either $c(T, \Sigma^{n+1}) \neq 0$ for $n \equiv 0 \pmod 4$, or $\sigma(T, \Sigma^{n+1}) \neq 0$ for $n \equiv 2 \pmod 4$. An interesting example to study in connection with the possibility of

[1] The authors were partially supported by contracts GP 2425 and GP 3685, respectively, with the National Science Foundation.

[2] The results hold equally in the piecewise linear category with little change in the proofs.

a nonzero Arf invariant is the nonstandard involution of Hirsch and Milnor on S^5, [3]. However, even if the Arf invariant is zero, our methods do not give a desuspension of this involution to the 4-sphere, because of the usual difficulties of finding a basis for $H_2(M^4)$ represented by imbedded spheres.

Regarding the uniqueness of the desuspension, we have

THEOREM 4. *If $n \geqq 4$ is even, and (T, Σ^{n+1}) desuspends to $(T|S_0^n, S_0^n)$ and to $(T|S_1^n, S_1^n)$, then $(T|S_0^n, S_0^n)$ and $(T|S_1^n, S_1^n)$ are equivariantly concordant in $\Sigma^{n+1}\times I$.*

We say that (T_0, S_0^n) and (T_1, S_1^n) are *concordant* if there exists a fixed point free involution $T: S^n\times I \to S^n \times I$, where $I=[0, 1]$, such that $T(S^n\times 0)=S^n\times 0$, and equivariant diffeomorphisms i_0: $(T_0, S_0^n) \to (T|S^n\times 0, S^n\times 0)$ and i_1: $(T_1, S_1^n)\to(T|S^n\times 1, S^n\times 1)$. If $\overline{T}$: $\Sigma^{n+1} \to \Sigma^{n+1}$ is a smooth, fixed point free involution, then (T_0, S_0^n) and (T_1, S_1^n) are *concordant in* $\Sigma^{n+1}\times I$ if they are concordant, and $(T, S^n\times I)$ is equivariantly imbedded in $(\overline{T}\times 1, \Sigma^{n+1}\times I)$ with $S^n\times 0\subset \Sigma^{n+1}\times 0$, $S^n\times 1 \subset \Sigma^{n+1}\times 1$. If $n>4$ is odd, the signature and Arf invariant, which appeared as obstructions to desuspending (T, Σ^k), now appear as obstructions to obtaining a concordance in $(T\times 1, \Sigma^{n+1}\times I)$ between two given desuspensions, $(T|S_0^n, S_0^n)$ and $(T|S_1^n, S_1^n)$. If S_0^{4k-1} and S_1^{4k-1} are two invariant spheres in (T, Σ^{4k}), then $\sigma(T, \Sigma^{4k}, S_0^{4k-1}, S_1^{4k-1})$, the signature of a certain bilinear form, is defined. We then have

THEOREM 5. *S_0^{4k-1} and S_1^{4k-1} are concordant in $(T\times 1, \Sigma^{4k}\times I)$ if and only if $\sigma(T, \Sigma^{4k}, S_0^{4k-1}, S_1^{4k-1})=0$. In particular, if $\sigma=0$, then $(T|S_0^{4k-1}, S_0^{4k-1})$ and $(T|S_1^{4k-1}, S_1^{4k-1})$ are equivariantly diffeomorphic.*

Now suppose S_0^{4k+1} and S_1^{4k+1} are invariant spheres in (T, Σ^{4k+2}). Then $c(T, \Sigma^{4k+2}, S_0^{4k+1}, S_1^{4k+1})$, an Arf invariant, is defined.

THEOREM 6. *S_0^{4k+1} and S_1^{4k+1} are concordant in $(T\times 1, \Sigma^{4k+2}\times I)$ if and only if $c(T, \Sigma^{4k+2}, S_0^{4k+1}, S_1^{4k+1})=0$.*

COROLLARY. *If $n\equiv 1$ (mod 4), there are at most two invariant n spheres in Σ^{n+1}, up to equivariant diffeomorphism.*

It is planned to present detailed proofs later. We will, however, indicate briefly some of the ideas involved.

2. **Characteristic submanifolds.** Let $T: \Sigma^{n+1}\to\Sigma^{n+1}$ be a fixed point free smooth involution. A *characteristic submanifold* $M^n\subset\Sigma^{n+1}$ is an n-manifold smoothly imbedded in Σ^{n+1} such that $\Sigma^{n+1}=A\cup TA$ with $A\cap TA=M^n$. We have a commutative square

$$\begin{array}{ccc} \Sigma^{n+1} & \longrightarrow & S^N \\ \downarrow \pi & & \downarrow \\ \Sigma^{n+1}/T & \xrightarrow{f} & P^N \end{array}$$

where N is large, P^N is a real projective N-space, and f classifies the principal Z_2-bundle $\Sigma^{n+1}\xrightarrow{\pi}\Sigma^{n+1}/T$. By making f transverse-regular [5] on P^{N-1}, $\pi^{-1}f^{-1}P^{N-1}$ will be a characteristic submanifold. It is easy to see that all characteristic submanifolds arise in this way. Any two characteristic submanifolds are equivariantly cobordant in $(T\times 1, \Sigma^{n+1}\times I)$. (The definition is analogous to that of concordance in $\Sigma^{n+1}\times I$.) It is this fact that makes the signature and Arf invariant independent of the choice of characteristic submanifold.

3. **The signature and Arf invariant.** Let M be a characteristic submanifold in Σ^{n+1}. Then $\Sigma^{n+1}=A\cup TA$ with $A\cap TA=M$. We have the Mayer-Vietoris sequence

$$\cdots\to H_{p+1}(\Sigma^{n+1})\to H_p(M)\xrightarrow{(i_A,\, i_{TA})} H_p(A)\oplus H_p(TA)\to H_p(\Sigma^{n+1})\to\cdots$$

If $n=2k$, $k>0$, and $p=k$, this becomes

$$0\to H_k(M^{2k})\xrightarrow{(i_A,\, i_{TA})} H_k(A)\oplus H_k(TA)\to 0$$

and so $H_k(M^{2k})=\ker\, i_A\oplus\ker\, i_{TA}$, and $T_*\ker\, i_A=\ker\, i_{TA}$. Since $M^{2k}\subset\Sigma^{2k+1}$, M is orientable, and a bilinear form $B(x, y)=x\cdot T_*y$ is defined, for x and y in $\ker\, i_A$. Since T preserves orientation in Σ^{2k+1}, it reverses orientation in M^{2k}, and the bilinear form B is symmetric (skew-symmetric) when the intersection form $x\cdot y$ is is skew-symmetric (symmetric). Therefore, given (T, Σ^{n+1}) and a characteristic submanifold M^n, if $n\equiv 2 \pmod 4$, the signature of the form $B(x, y)$ is determined, and turns out to be independent of the choice of characteristic submanifold. The reason for considering the signature of B is the following. If $x\in\ker\, i_A\subset H_k(M^{2k})$, and M^{2k} is $(k-1)$-connected, (which we achieve by exchanging handles between A and TA) then x is represented by an imbedded $S^k\subset M^{2k}$, which bounds a cell $D^{k+1}\subset A$. (This statement may be false for $k=3$, [2], but a different argument applies in this case.) Supposing M^{2k} is totally geodesic near D^{k+1}, we take a tubular neighborhood N of D^{k+1}, replace A by $A-N$, and replace TA by $TA\cup\overline{N}$. This will reduce the rank of $H_k(M)$. However, $(A-N)\cap(TA\cup\overline{N})=M'$ is no longer T-invariant. We may obtain an invariant M' if we replace A by $(A-N)\cup T\overline{N}$, and replace TA by $(TA\cup\overline{N})-TN$. However, to do this we need $S^k\cap TS^k=\phi$. It is to accomplish this that we need $\sigma=0$

when k is odd and $c=0$ when k is even. The distinction between the two cases arises since if S^k and TS^k intersect transversally in M^{2k} at a point p with intersection number 1, then they intersect at Tp with intersection number $(-1)^{k+1}$.

The cohomology operation, $\psi(x)$, used to define the Arf invariant, merely serves to count, mod 2, the number of pairs (q, Tq) of points in $S^p \cap TS^p$, where S^p represents the Poincaré dual of x, and the intersection is transverse.

References

1. C. Arf, *Untersuchungen über quadratische Formen in Körpern der Charakteristik* 2, J. Reine Angew. Math. **183** (1941), 148–167.

2. A. Haefliger, *Knotted* $(4k-1)$-*spheres in* 6*k*-*space*, Ann. of Math. **75** (1962), 452–466.

3. M. W. Hirsch and J. W. Milnor, *Some curious involutions of spheres*, Bull. Amer. Math. Soc. **70** (1964), 372–377.

4. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*. I, Ann. of Math. **77** (1963), 504–537.

5. R. Thom, *Quelques propriétés global des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

Institute for Advanced Study,
Princeton University and
Cornell University