REFLEXION SPACES AND HOMOGENEOUS
SYMMETRIC SPACES

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1. A reflexion space is a set \( M \) with a multiplication \( \mu : M \times M \rightarrow M \),
\((x, y) \mapsto x \cdot y\), satisfying the following axioms:

\begin{align*}
(S_1) & \quad x \cdot x = x, \\
(S_2) & \quad x \cdot (x \cdot y) = y, \\
(S_3) & \quad x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z).
\end{align*}

Let be \( S(x) : y \mapsto x \cdot y \) the left multiplication with \( x \) in \( M \). This is an
involutive map of \( M \) onto itself leaving \( x \) fixed, which may be interpreted as the reflexion in the point \( x \).

Let \( \mathfrak{A} \) be a finite dimensional Jordan algebra and \( I \) the set of invertible elements of \( \mathfrak{A} \). In general for \( x, y \in I \) their product \( xy \) is not
in \( I \), so \( I \) does not inherit a multiplicative structure from \( \mathfrak{A} \). However, \( x \cdot y = 2x(xy^{-1}) - x^2y^{-1} \) is invertible ([1]), and the multiplication
\( x \cdot y \) makes \( I \) a reflexion space. Every group is a reflexion space with
the new product \( x \cdot y = xy^{-1}x \). Every set is a reflexion space with the
trivial product \( x \cdot y = y \) for all \( x \) and \( y \).

A reflexion space \( M \) where \( M \) is a connected paracompact \( C^\infty \)
manifold and \( \mu : M \times M \rightarrow M \) is differentiable is called a differentiable
reflexion space. The following construction gives examples. Let \( G \)
be a connected Lie group, \( \sigma \) an involutive automorphism of \( G \) and \( H \)
a subgroup of \( G \) lying between the group of all fixed points of \( \sigma \) and
its identity component. Then \( G/H \) is a homogeneous symmetric space
and \( G(G/H, H) \) is a principal fibre bundle with base space \( G/H \) and
structure group \( H \). Let \( H \) operate on a connected manifold \( F \) on the
left and let be \( G \times_H F \) the bundle associated with \( G(G/H, H) \) with
typical fibre \( F \) (cf [2]). We denote the equivalence class of \((g, x) \in G \times F \) in
\( G \times H F \) by \( g \otimes x \). In case \( F \) is a point, we have \( G \times_H F = G/H \).

Proposition 1. \( G \times_H F \) is a differentiable reflexion space with the
multiplication

\[(f \otimes x) \cdot (g \otimes y) = (f(f^*)^{-1}g^*) \otimes y.\]

¹ This work is a generalization of part of the author's doctoral dissertation at the
University of Munich under Professor M. Koecher.

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Our main result is that all differentiable reflexion spaces are of the type described above (Theorem 2). Proofs will be given in a forthcoming article.

2. A differentiable map $\phi: M \to N$ of differentiable reflexion spaces $M, N$ is called a homomorphism, if $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$. Isomorphisms and automorphisms are defined as usual. Let $\text{Aut} M$ be the group of (differentiable) automorphisms of $M$ and $G$ the group generated by the maps $S(x) \circ S(y)$, $x, y \in M$. We denote the image of a tangent vector $X$ of $M$ under the differential of the map $x \mapsto p \cdot x$ by $p \cdot X$; similarly $X \cdot p$ is defined. A vector $X \in T_p(M)$ is called vertical (horizontal), if $p \cdot X = X$ (resp. $p \cdot X = -X$). The set $T^+(M)$ of vertical vectors (resp. $T^-(M)$ of horizontal vectors) form subbundles of the tangent bundle $T(M)$ and we have $T(M) = T^+(M) \oplus T^-(M)$. Let $J$ be the projection on $T^-(M)$ and


the Nijenhuis torsion tensor of $J$.

**Proposition 2.** $T^+(M)$ is an involutive subbundle of $T(M)$. $T^-(M)$ is involutive if and only if $S = 0$.

Let $F_p$ be the maximal connected integral manifold of $T^+(M)$ through $p$, called the fibre through $p$. It can be characterized as the connected component of $p$ of the set of fixed points of $S(p)$. The set $M_0$ of all fibres becomes a reflexion space with the multiplication $F_p \cdot F_q = F_{p \cdot q}$.

3. Let $M$ be a differentiable reflexion space, $e$ a fixed point of $M$ and $F = F_e$, the fibre through $e$. For $X \in T_e(M)$ let $L(X)$ be the vector field given by

$$L(X)_p = \frac{1}{2} X \cdot (e \cdot p),$$

and $\mathfrak{g}$ the Lie algebra generated by the vector fields $L(X)$. Let $G$ denote the group generated by the transformations $S(x) \circ S(y)$, $x, y \in M$.

**Theorem 1.** $G$ has a unique structure of a connected Lie transformation group of $M$, so that its Lie algebra is isomorphic with $\mathfrak{g}$. It permutes the fibres transitively. The map $\sigma: \mathfrak{g} \to S(e) \circ g \circ S(e)$ is an involutive automorphism of $G$ and the subgroup $H$ of $G$ which leaves $F$ invariant lies between the group of fixed points of $\sigma$ and its identity component.

We now state our main result.
Theorem 2. $M$ is isomorphic as a reflexion space with $G \times H F$ under the map $g \otimes x \mapsto g(x) \ (g \in G, x \in F)$. The set of fibres $M_0$ is isomorphic with $G/H$ under $gH \mapsto F_{g(e)}$, and the diagram

$$
\begin{array}{ccc}
G \times H F & \rightarrow & M \\
\downarrow & & \downarrow \\
G/H & \rightarrow & M_0
\end{array}
$$

is commutative. Moreover, $M_0$ depends functorially on $M$.

4. We keep the notation of the preceding sections. A differentiable reflexion space $M$ is called torsion free, if $S = 0$. The maximal connected integral manifolds of $T^{-}(M)$ (Proposition 2) are called the leaves of $M$.

Proposition 3. $S = 0$ if and only if the identity component $H^0$ of $H$ operates trivially on $F$.

By a result of Koh [3] $H/H^0$ is finite. Let be $\Gamma$ the finite group of diffeomorphisms of $F$ induced by $H/H^0$. A point of $F$ is called regular if its isotropy subgroup in $\Gamma$ consists of the identity alone, singular otherwise. For the notion of a regular leaf see [4].

Theorem 3. Let $M$ be torsion free. The leaves are closed submanifolds and coincide with the orbits of $G$. A leaf is regular if and only if it intersects $F$ in a regular point. The restriction of the canonical projection $M \rightarrow M_0$ to a leaf $B$ is a finite covering map $B \rightarrow M_0$, which is regular with group $\Gamma$ if $B$ is regular. If $M_0$ is simply connected, then $M$ is isomorphic with $M_0 \times F$, where $F$ has the trivial multiplication (see §1).

A vector field $X$ on $M$ is called a derivation, if

$$X_{p^*q} = X_p^*q + p^*X_q$$

for $p, q \in M$. The set of derivations is a Lie algebra $\text{Der} M$. In general the automorphism group $\text{Aut} M$ is too big to be a Lie group. However we have

Theorem 4. Let $G$ be transitive on $M$. Then $\text{Aut} M$ has a unique structure of a Lie transformation group of $M$ so that its Lie algebra is isomorphic with $\text{Der} M$.

References
