TWO-SIDED IDEALS IN C^*-ALGEBRAS

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If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{J}$ and $\mathfrak{F}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then so is $\mathfrak{J}+\mathfrak{F}$. The following problem has been proposed by J. Dixmier [1, Problem 1.9.12]: is $(\mathfrak{J}+\mathfrak{F})^+=\mathfrak{J}^++\mathfrak{F}^+$, where $\mathfrak{K}^+$ denotes the set of positive operators in a family $\mathfrak{K}$ of operators? He suggested to the author that techniques using the duality between invariant faces of the state space $S(\mathfrak{A})$ of $\mathfrak{A}$ and two-sided ideals in $\mathfrak{A}$, as shown by E. Effros, might be helpful in studying it. In this note we shall use such arguments to solve the problem to the affirmative.

By a *face* of $S(\mathfrak{A})$ we shall mean a convex subset F such that if $\rho\in F$, $\omega\in S(\mathfrak{A})$ and $a\omega\leqq\rho$ for some $a>0$, then $\omega\in F$. F is an *invariant face* if $\rho\in F$ implies the state $B\rightarrow\rho(A^*BA)\cdot\rho(A^*A)^{-1}$ belongs to F whenever $\rho(A^*A)\neq 0$ and $A\in\mathfrak{A}$. We denote by $F^\perp$ the set of operators $A\in\mathfrak{A}$ such that $\rho(A)=0$ for all $\rho\in F$. If $\mathfrak{J}\subset\mathfrak{A}$, $\mathfrak{J}^\perp$ shall denote the set of states ρ such that $\rho(A)=0$ for all $A\in\mathfrak{J}$. E. Effros [2] has shown that the map $\mathfrak{J}\rightarrow\mathfrak{J}^\perp$ is an order inverting bijection between uniformly closed two-sided ideals of $\mathfrak{A}$ and w^*-closed invariant faces of $S(\mathfrak{A})$. Moreover, $(\mathfrak{J}^\perp)^\perp=\mathfrak{J}$, and $(F^\perp)^\perp=F$ when F is a w^*-closed invariant face. If $\mathfrak{J}$ and $\mathfrak{F}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then $(\mathfrak{J}\cap\mathfrak{F})^\perp=\operatorname{conv}(\mathfrak{J}^\perp,\mathfrak{F}^\perp)$, the convex hull of $\mathfrak{J}^\perp$ and $\mathfrak{F}^\perp$, and $(\mathfrak{J}+\mathfrak{F})^\perp=\mathfrak{J}^\perp\cap\mathfrak{F}^\perp$. If A is a self-adjoint operator in $\mathfrak{A}$ let $\hat{A}$ denote the w^*-continuous affine function on $S(\mathfrak{A})$ defined by $\hat{A}(\rho)=\rho(A)$. It has been shown by R. Kadison, [3] and [4], that the map $A\rightarrow\hat{A}$ is an isometric order-isomorphism of the self-adjoint part of $\mathfrak{A}$ onto all w^*-continuous real affine functions on $S(\mathfrak{A})$. Moreover, if $\mathfrak{J}$ is a uniformly closed two-sided ideal in $\mathfrak{A}$, and ψ is the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$, then the map $\rho\rightarrow\rho\circ\psi$ is an affine isomorphism of $S(\mathfrak{A}/\mathfrak{J})$ onto $\mathfrak{J}^\perp$. Thus the map $\psi(A)\rightarrow\hat{A}\,|\,\mathfrak{J}^\perp$ is an order-isomorphic isometry on the self-adjoint operators in $\mathfrak{A}/\mathfrak{J}$. We shall below make extensive use of these facts. For other references see [1, §1].

THEOREM. *Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{J}$ and $\mathfrak{F}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then*

$$(\mathfrak{J}+\mathfrak{F})^+=\mathfrak{J}^++\mathfrak{F}^+.$$

In order to prove the theorem we may assume $\mathfrak{A}$ has an identity, denoted by I. We first prove a

LEMMA. *With the assumptions as in the theorem let A belong to $(\mathfrak{J}+\mathfrak{F})^+$, and let $\epsilon>0$ be given, $\epsilon<1$. Then there exist B in $\mathfrak{J}^+$ and C in $\mathfrak{F}^+$ such that $0\leqq A-B-C\leqq \epsilon I$.*

PROOF. We may assume $\|A\|\leqq 1$. Let ψ denote the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{F}$. Then $\psi(\mathfrak{J}+\mathfrak{F})=\psi(\mathfrak{J})$. Now $\psi(A)\geqq 0$. Therefore there exists $B_1\in\mathfrak{J}^+$ such that $\psi(B_1)=\psi(A)$. Then $\hat{B}_1|\mathfrak{J}^\perp=0$ and $\hat{B}_1|\mathfrak{F}^\perp=\hat{A}|\mathfrak{F}^\perp$. Since $(\mathfrak{J}\cap\mathfrak{F})^\perp=\text{conv}(\mathfrak{J}^\perp,\ \mathfrak{F}^\perp)$, $\hat{B}_1|(\mathfrak{J}\cap\mathfrak{J})^\perp\leqq\hat{A}|(\mathfrak{J}\cap\mathfrak{F})^\perp$. Let ϕ denote the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}\cap\mathfrak{F}$. Then $0\leqq\phi(B_1)\leqq\phi(A)$. Let f be the real continuous function $f(x)=(\epsilon/3)^2$ for $x\leqq(\epsilon/3)^2$, $f(x)=x$ for $x>(\epsilon/3)^2$. Let

$$S = f(A)^{-1/2}B_1f(A)^{-1/2}.$$

Then $S\in\mathfrak{J}^+$, and

$$\begin{aligned} 0 \leqq \phi(S) &= f(\phi(A))^{-1/2}\phi(B_1)f(\phi(A))^{-1/2} \\ &\leqq f(\phi(A))^{-1/2}\phi(A)f(\phi(A))^{-1/2} \\ &\leqq \phi(I). \end{aligned} \tag{1}$$

Let g be the real continuous function $g(x)=x$ for $x\leqq 1$, $g(x)=1$ for $x>1$. Since $g(0)=0$, $g(S)$ is by the Stone-Weierstrass theorem a uniform limit of polynomials in S without constant terms. Since $S\in\mathfrak{J}^+$, and $\mathfrak{J}$ is uniformly closed, $g(S)\in\mathfrak{J}^+$. By (1)

$$\phi(g(S)) = g(\phi(S)) = \phi(S). \tag{2}$$

Let

$$B = (f(A)^{1/2} - (\epsilon/3)I)g(S)(f(A)^{1/2} - (\epsilon/3)I).$$

Since $g(S)\in\mathfrak{J}^+$ so is B. Now $(f(x)^{1/2}-\epsilon/3)^2\leqq x$ for $x\geqq 0$, and $g(S)\leqq I$. Hence $0\leqq B\leqq A$. By (2)

$$\begin{aligned} \phi(B) &= (f(\phi(A))^{1/2} - (\epsilon/3)\phi(I))\phi(g(S))(f(\phi(A))^{1/2} - (\epsilon/3)\phi(I)) \\ &= \phi(B_1) - (\epsilon/3)\left[f(\phi(A))^{1/2}\phi(S) + \phi(S)f(\phi(A))^{1/2} - (\epsilon/3)\phi(S)\right]. \end{aligned}$$

Since $\|f(\phi(A))^{1/2}\|\leqq 1$, $\|\phi(S)\|\leqq 1$, and $\epsilon<1$

$$\|\hat{B}\,|\,(\mathfrak{J}\cap\mathfrak{F})^\perp - \hat{B}_1\,|\,(\mathfrak{J}\cap\mathfrak{F})^\perp\| = \|\phi(B) - \phi(B_1)\| \leqq \epsilon.$$

In particular,

$$\|\hat{B}\,|\,\mathfrak{F}^\perp - A\,|\,\mathfrak{F}^\perp\| = \|\hat{B}\,|\,\mathfrak{F}^\perp - \hat{B}_1\,|\,\mathfrak{F}^\perp\| \leqq \epsilon. \tag{3}$$

Apply the preceding to $A-B$ instead of A and to $\mathfrak{F}$ instead of $\mathfrak{J}$. Choose $C_1\in\mathfrak{F}^+$ such that $C_1\leqq A-B$, and

$$\|C_1\,|\,\mathfrak{J}^\perp - (A - B)\,|\,\mathfrak{J}^\perp\| \leqq \epsilon. \tag{4}$$

Since $\hat{C}_1 | \mathfrak{F}^\perp = 0$, (3) implies

$$\|\hat{C}_1 | \mathfrak{F}^\perp - (A - \hat{B}) | \mathfrak{F}^\perp\| \leqq \epsilon. \tag{5}$$

By (4) and (5)

$$\begin{aligned}\|\phi(C_1) - \phi(A - B)\| = \|\hat{C}_1 | \operatorname{conv}(\mathfrak{J}^\perp, \mathfrak{F}^\perp) \\ - (A - \hat{B}) | \operatorname{conv}(\mathfrak{J}^\perp, \mathfrak{F}^\perp)\| \leqq \epsilon.\end{aligned}$$

Let $D = A - (B + C_1)$. Then $D \geqq 0$, and $\|\phi(D)\| \leqq \epsilon$. Let h be the real continuous function $h(x) = 0$ for $x \leqq \epsilon$, $h(x) = x - \epsilon$ for $x > \epsilon$. Then $\phi(h(D)) = h(\phi(D)) = 0$, and $h(D) \in (\mathfrak{J} \cap \mathfrak{F})^+ \subset \mathfrak{F}^+$. Furthermore

$$D - \epsilon I \leqq h(D) \leqq D. \tag{6}$$

Let $C = C_1 + h(D)$. Then $C \in \mathfrak{F}^+$, and by (6)

$$0 \leqq B + C \leqq B + C_1 + D = A \leqq B + C_1 + h(D) + \epsilon I = B + C + \epsilon I.$$

The proof is complete.

PROOF OF THEOREM. Let $A \in (\mathfrak{J} + \mathfrak{F})^+$. Multiplying A by a scalar we may assume $0 \leqq A \leqq I$. By the lemma choose $B_0 \in \mathfrak{J}^+$, $C_0 \in \mathfrak{F}^+$ such that

$$0 \leqq A - B_0 - C_0 \leqq 2^{-1} I.$$

Then $\|B_0\| \leqq \|A\| \leqq 1$, $\|C_0\| \leqq \|A\| \leqq 1$. Suppose inductively $B_0, B_1, \cdots, B_{n-1}$ are chosen in $\mathfrak{J}^+$ and $C_0, C_1, \cdots, C_{n-1}$ are chosen in $\mathfrak{F}^+$ such that $\|B_j\| \leqq 2^{-j}$, $\|C_j\| \leqq 2^{-j}$, and

$$0 \leqq A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \leqq 2^{-n} I.$$

Apply the lemma to $A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j$ and to $\epsilon = 2^{-n-1}$. Then there exist $B_n \in \mathfrak{J}^+$, $C_n \in {}^+\mathfrak{F}$ such that

$$0 \leqq A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j - B_n - C_n \leqq 2^{-n-1} I, \tag{7}$$

or

$$0 \leqq A - \sum_{j=0}^{n} B_j - \sum_{j=0}^{n} C_j \leqq 2^{-n-1} I.$$

Moreover, by (7) $\|B_n\| \leqq 2^{-n}$, $\|C_n\| \leqq 2^{-n}$; the induction argument is complete. Let

$$B = \sum_{j=0}^{\infty} B_j, \qquad C = \sum_{j=0}^{\infty} C_j.$$

Then $B \in \mathfrak{J}^+$, $C \in \mathfrak{J}^+$, and

$$\|A - B - C\| = \lim_{n \to \infty} \left\|A - \sum_{j=0}^{n} B_j - \sum_{j=0}^{n} C_j\right\| \leqq \lim_{n \to \infty} 2^{-n-1} = 0.$$

Thus $A = B + C \in \mathfrak{J}^+ + \mathfrak{F}^+$, and $(\mathfrak{J} + \mathfrak{F})^+ \subset \mathfrak{J}^+ + \mathfrak{F}^+$. Since the converse inclusion is trivial, the proof is complete.

References

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