RESEARCH ANNOUNCEMENTS

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PRINCIPAL PARTITIONS AND GENERATORS

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0. **Introduction.** In [1] Rohlin proved that every aperiodic metric automorphism of a Lebesgue space with finite entropy possesses an a generator with finite entropy and in [2] he proved that every aperiodic countable to one metric endomorphism possesses a countable generator. This latter problem was also solved by the present author for the case of an algebraic automorphism [3]. In this note we present alternative approaches to these theorems. §2 is concerned with metric automorphisms and Rohlin's theorem is established in a manner which exhibits the relationship between principal partitions and a generators. §3 is concerned with algebraic endomorphisms and Rohlin's theorem is established by means of an adaptation of the method of [3]. Details will be published elsewhere; it is hoped that the sequence in which the results are presented will illustrate the methods employed.

For an account of the definitions and basic properties of Lebesgue spaces, measurable partitions, metric homomorphisms, endomorphisms and automorphisms cf. [4]. For the definitions and basic properties of conditional entropy $H(\xi/\eta)$ and entropy $H(\xi)$ defined for measurable partitions ξ, η cf. [4], [5].

An algebraic homomorphism [endomorphism, automorphism] is defined in the same way as a metric homomorphism [endomorphism, automorphism] except that the measure preserving property is replaced by the condition of nonsingularity.

Throughout we will be concerned with ergodic endomorphisms T of a Lebesgue space (X, B, m). (The method we outline could easily be adapted to the case of aperiodic endomorphisms.) Equations involving partitions or sets will be interpreted mod 0.

1. **Notations and definitions.** Z denotes the set of all measurable partitions ξ such that $H(\xi) < \infty$. (If $\xi \in Z$ then ξ is countable mod 0.)

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We use the notation $\xi^- = \bigvee_{i=0}^{\infty} T^{-i}\xi$ and when T is an automorphism $\xi_T = \bigvee_{i=-\infty}^{\infty} T^i\xi$. ϵ denotes the unit partition of X into single point sets.

If $\xi^- = \epsilon$ then ξ is called a generator [2]. (In [3] such a partition ξ was called a strong generator.) If T is an automorphism and $\xi_T = \epsilon$, then ξ is called an *a* generator [2]. (In [1] such a partition ξ was called a generator.)

For a metric endomorphism T we let $h(T, \xi) = H(\xi/\xi^-)$ and $h(T) = \sup_{\xi \in Z} h(T, \xi)$. This latter quantity is called the entropy of T.

We shall need repeatedly the following basic lemma of Rohlin [1], [6].

LEMMA R. *If $\alpha \geqq \beta$ and the elements of β comprise a countable number of elements of α* (mod 0) *then there exists a countable partition γ such that $\alpha = \beta \vee \gamma$. This happens in particular when $H(\alpha/\beta) < \infty$ and in this case γ can be chosen so that $H(\gamma) < \infty$. When $H(\alpha/\beta) \leqq \frac{1}{4}$ then γ can be chosen so that $H(\gamma) \leqq 15(H(\alpha/\beta))^{1/2}$.*

COROLLARY. *If T is an ergodic automorphism and $T\alpha = \alpha$, $T\beta = \beta$ where the elements of β comprise a countable number of elements of α* (mod 0) *(in particular if $H(\alpha/\beta) < \infty$) then $\alpha = \beta \vee \gamma_1^-$ if $\gamma_1 = (G_1, X - G)$ where $m(G_1) > 0$ and $G_1 \subset G \in \gamma$. Consequently for every $\delta > 0$ there exists $\gamma_1 \in Z$, $H(\gamma_1) < \delta$, such that $\alpha = \beta \vee \gamma_1^-$.*

2. *a* **generators for ergodic metric automorphisms with finite entropy.** Throughout this section T will denote an ergodic metric automorphism with finite entropy.

η is said to be a *principal* partition if $T\eta = \eta$ and $h(T) = h(T_\eta)$ or equivalently, if $T\eta = \eta$ and $T\zeta \geqq \zeta \geqq \eta$ implies $T\zeta = \zeta$ [7].

LEMMA 1. *If $T\zeta \geqq \zeta$ and $h(T) = H(\zeta/T^{-1}\zeta)$ (such partitions can be constructed by the methods of [8]), then $\zeta = \xi \vee T^{-1}\zeta$ where $\xi \in Z$. For any such ξ, ξ_T is principal.*

The proof depends on Lemma R.

COROLLARY. *There exists $\xi \in Z$ such that ξ_T is principal.*

LEMMA 2. *If η is principal and $\delta > 0$ then there exists $\gamma \in Z$, $H(\gamma) < \delta$, such that $\gamma_T \vee \eta = \epsilon$.*

The proof is based on Lemma R and its corollary.

Lemma 2 and the corollary to Lemma 1 imply:

THEOREM 1 [1]. *There exists an a generator $\alpha \in Z$. Moreover the set of a generators is dense in $\{\xi \in Z\colon h(T) = h(T, \xi)\}$. (Here Z is endowed with the metric $d(\alpha, \xi) = H(\alpha/\xi) + H(\xi/\alpha)$.)*

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3. **Generators for ergodic countable to one algebraic endomorphisms.** In this section T denotes an ergodic countable to one algebraic endomorphism.

LEMMA 3. *There exists a countable partition β and a sequence of finite partitions $\epsilon_1 \leqq \epsilon_2 \leqq \cdots \epsilon_n \uparrow \epsilon$ such that $\epsilon = \beta \vee T^{-1}\epsilon$ and $\beta \vee T^{-1}\epsilon_n \geqq \epsilon_{n-1}$.*

Lemma 3, which rests on Lemma R, permits an adaptation of the method of [3] to prove:

THEOREM 2 [2]. *There exists a countable generator.*

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