EXISTENCE AND PERTURBATION THEOREMS FOR NONLINEAR MAXIMAL MONOTONE OPERATORS IN BANACH SPACES

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Let X be a Banach space, X^* its conjugate space with the pairing between w in X^* and u in X denoted by (w, u). If T is a mapping (in general, nonlinear) with domain $D(T)$ in X and range $R(T)$ in X^*, T is said to be monotone if for all u and v of $D(T)$

$$(1) \qquad (T(u) - T(v), u - v) \geqq 0.$$

More generally, a subset G of $X \times X^*$ is said to be a monotone set if for each pair $[u_1, w_1]$ and $[u_2, w_2]$ in G, we have

$$(2) \qquad (w_2 - w_1, u_2 - u_1) \geqq 0.$$

Such a set G is said to be maximal monotone if it is maximal among monotone sets in the sense of set inclusion, and a mapping T is said to be maximal monotone if its graph $G(T)$ is a maximal monotone set.

For reflexive Banach spaces X and mappings T with $D(T) = X$, the basic result obtained independently by Browder [2] and Minty [20] states that if T is a monotone operator from $D(T) = X$ to X^* which is *hemicontinuous*, (i.e. continuous from each line segment in $D(T)$ to the weak topology on X^*), and *coercive*, i.e.

$$(3) \qquad (Tu, u)/\|u\| \to + \infty \quad (\text{as } \|u\| \to + \infty)$$

then the range $R(T)$ of T is the whole of the space X^*. This theorem and its extensions to various classes of operators T from all of X to X^* which satisfy modified monotonicity conditions (Browder [4], [9], [10], [11], [12], Leray-Lions [15], Hartman-Stampacchia [14]) are the basis of the application of the theory of monotone operators to obtain general existence theorems for nonlinear elliptic boundary value problems.

For nonelliptic problems, the corresponding reduction to equations for nonlinear operators acting from a reflexive Banach space X to its dual space X^* yields operators T which are only defined on a dense subset $D(T)$ of X. Results covering the principal parabolic and hyperbolic problems were obtained in Browder [3], [5], [7] for operators T of the form $T = L + T_0$, where T_0 is an everywhere defined hemicontinuous monotone nonlinear operator from X to X^* which is coercive and maps bounded subsets of X into bounded sub-

sets of X^* while L is a closed densely-defined monotone linear operator from X to X^* such that its adjoint L^* is the closure of its restriction to $D(L) \cap D(L^*)$.

It is the purpose of the present note to present some new and much more general results on maximal monotone operators from a reflexive Banach space X to its dual space X^* which permit a wider and more flexible application of the theory of densely defined monotone operators. The detailed proofs appear in [13], but we shall develop the principal tool of the argument in Proposition 1 below.

Our basic result is the following.

THEOREM 1. *Let X be a reflexive Banach space, T_1 a maximal monotone mapping from X to X^* with $0 \in D(T_1)$. Let T_2 be a hemicontinuous monotone mapping from all of X to X^* which is coercive and maps bounded subsets of X into bounded subsets of X^*. Let $T = T_1 + T_2$, with $D(T) = D(T_1)$.*
Then the range $R(T)$ of T is all of X^*.

Some important consequences of Theorem 1 are the following:

THEOREM 2. *Let X be a strictly convex reflexive Banach space with a strictly convex conjugate space X^*, T_1 a maximal monotone mapping from X to X^*, T_2 a hemicontinuous monotone mapping of all of X into X^* which carries bounded subsets of X into bounded subsets of X^*.*
Then the mapping $T = T_1 + T_2$ is a maximal monotone map of X into X^*.

THEOREM 3. *Let X be a reflexive Banach space with a strictly convex dual space and suppose that T is a coercive maximal monotone mapping from X to X^* with $0 \in D(T)$.*
Then $R(T)$ is all of X^*.

THEOREM 4. *Let X be a uniformly convex Banach space with a strictly convex dual space X^*, T a maximal monotone mapping from X to X^*. Suppose that the inverse image under T of each bounded set is bounded.*
Then $R(T)$ is all of X^*.

The strict convexity and uniform convexity hypotheses of Theorems 2, 3, and 4 can be avoided (as we shall show in another paper) by systematic use of multivalued mappings.

It is shown in [13] that Theorem 1, as well as Theorems 2, 3, and 4, are valid for T_1 and T multivalued mappings from X to X^*. This is of particular importance in the application of these results to the subclass of the monotone mappings which consists of the subdifferentials of proper lower semicontinuous convex functions f from X to

$R^1 \cup \{ + \infty \}$, ([1], [23], [24]). An element w of X^* is said to be a subgradient of f at u in X provided that for all v in X

$$(4) \qquad\qquad f(v) \geq f(u) + (w, v - u).$$

The subdifferential $\partial f(u)$ is the set of all subgradients w for f at u. Rockafellar [24] has shown that for any Banach space X and any proper l.s.c. convex function f on X, ∂f is a maximal monotone (in general, multivalued) mapping from X to X^*.

THEOREM 5. *Let X be a reflexive Banach space with X^* strictly convex, f a proper lower semicontinuous convex function on X, ∂f its subdifferential. Suppose that $0 \in D(\partial f)$, and let J be a duality mapping of X into X^*.*

Then (a) *The mapping $T = \partial f + J$ maps onto X^*,*

(b) *If X is uniformly convex, $(\partial f + J)^{-1}$ is a continuous mapping of X^* onto $D(\partial f)$. In particular, $D(\partial f)$ is pathwise-connected.*

For X a Hilbert space H, the result of Theorem 5 is due to Moreau [23], and is related to a theorem of Minty [19] on maximal monotone sets in Hilbert space. For the case treated here, Theorem 5 answers a question posed by Rockafellar in [24].

The proofs of our theorems combine ideas from the study of non-linear variational inequalities (Browder [11], [12]) and a new approach to the theory of monotone operators on convex sets developed by Minty [22]. (Earlier results on monotone inequalities on convex sets and linear variational inequalities are given in [25], [8], [14], [10], [16], [17], [18]). In particular, Minty proves the following (Theorem 3 of [22]).

LEMMA. *Let K be a compact convex subset of a locally convex topological vector space E, G a monotone subset of $K \times E^*$, w_0 an arbitrary element of E^*. Then there exists an element u_0 of K such that:*

$$(5) \qquad (w - w_0, u - u_0) \geq 0, \quad (for\ all\ [u, w]\ in\ G).$$

We use the following sharper result, which combines Minty's lemma, Lemma 2 of [8], and Lemma 4 of [11]. Its proof, which is based upon that of Lemma 4 of [11], gives an alternative proof of a different type for Minty's lemma.

PROPOSITION 1. *Let K be a compact convex subset of a locally convex space E, G a monotone subset of $K \times E^*$, T a continuous map of K into E^*. Let w_0 be an element of E^*.*

Then there exists u_0 in K such that for all $[u, w]$ in G,

$$(6) \qquad\qquad (T(u_0) + w - w_0, u - u_0) \geq 0.$$

PROOF OF PROPOSITION 1. We may assume without loss of generality that $w_0 = 0$. Suppose that the assertion of Proposition 1 is false. Then for each x in K, there exists at least one element $[u, w]$ in G with

$$(7) \qquad (T(x) + w, u - x) < 0.$$

For each element $[u, w]$ in G, let $N_{u,w}$ be the open set in K given by

$$(8) \qquad N_{u,w} = \{x \mid x \in K, (T(x) + w, u - x) < 0\}.$$

The family $\{N_{u,w}: [u, w] \in G\}$ is then an open covering of the compact set K, and hence there exists a finite family $\{[u_j, w_j], 1 \leq j \leq n\}$ in G such that the corresponding N_{u_j, w_j} cover K. We form a continuous partition of unity on K corresponding to this finite open covering, i.e. an n-tuple $\{\beta_1, \cdots, \beta_n\}$ of continuous real functions on K such that for each j, β_j vanishes outside of N_{u_j, w_j}, $0 \leq \beta_j(x) \leq 1$, and for all x in K, $\sum_{j=1}^{n} \beta_j(x) = 1$.

We define two continuous mappings p and q of K into E and E^*, respectively, by:

$$(9) \qquad p(x) = \sum_{j=1}^{n} \beta_j(x) u_j, \qquad q(x) = \sum_{j=1}^{n} \beta_j(x) w_j.$$

For each x in K, $p(x)$ is a convex linear combination of points of K and hence lies in K. Therefore p is a continuous self-mapping of K, and by the Tychonoff fixed point theorem, p has a fixed point x_0 in K, $(p(x_0) = x_0)$.

On the other hand, if we let

$$(10) \qquad \lambda(x) = (T(x) + q(x), x - p(x)), \qquad (x \in K),$$

and expand p and q, we find that $\lambda(x) = \lambda_1(x) + \lambda_2(x)$, where

$$\lambda_1(x) = \sum_{j=1}^{n} \beta(x)^2 (T(x) + w_j, x - u_j) > 0,$$

for each x in K, while:

$$\lambda_2(x) = \sum_{1 \leq j < k \leq n} \beta_j(x) \beta_k(x) [(T(x) + w_j, x - u_k) + (T(x) + w_k, x - u_j)].$$

We note that:

$$(w_j, x - u_k) + (w_k, x - u_j) = (w_j, x - u_j) + (w_k, x - u_k)$$
$$+ (w_j - w_k, u_j - u_k).$$

Hence for all x in K,

$$\lambda_2(x) = \sum_{j<k} \beta_j(x)\beta_k(x) \big[(T(x) + w_j, x - u_j)$$

$$+ (T(x) + w_k, x - u_k)$$

$$+ (w_j - w_k, u_j - u_k) \big] \geqq 0.$$

Finally, $\lambda(x) = \lambda_1(x) + \lambda_2(x) > 0$ for all x in K, while

$$\lambda(x_0) = (T(x_0) + q(x_0), x_0 - p(x_0)) = 0.$$

This contradiction proves the proposition. q.e.d.

The writer has noted that Theorem 5 can also be obtained as a corollary of the results on nonlinear variational inequalities given in Browder [11]. The methods presented here (and developed in detail in [13]) can also be applied to nonlinear variational inequalities involving maximal monotone operators (as well as maximal monotone operators on convex sets) and to more general classes of operators satisfying monotonicity conditions of weaker types.

Added in proof. The writer has discovered that Proposition 1 (and thereby Minty's lemma as well) was proved by H. Debrunner and P. Flor, Archiv. Math. **15** (1964), 445–447, using fixed point theorems for multivalued mappings. The method of proof given above has been applied by the writer to obtain more general fixed point theorems for single-valued and multivalued mappings in topological vector spaces.

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