MULTIPLICATIVE FIBRE MAPS

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In this note we shall outline a result concerning the cohomology of a multiplicative fibre map. To fix our notation we shall assume that

\[ F \xrightarrow{i} E \xrightarrow{\pi} B \]

is a Serre fibre space such that

1. \( F, E, B \) are \( H \)-spaces (homotopy associative) and \( F \to E, E \to B \) are \( H \)-maps.
2. \( B \) is simply connected.
3. \( H^*(B; \mathbb{Z}_p) \) is a polynomial algebra, where \( \mathbb{Z}_p \) denotes the integers modulo \( p \), \( p \) a prime.
4. \( H_*(B; \mathbb{Z}_p) \) is a commutative algebra.

The result that we shall establish is

**THEOREM.** If \( H^*(E; \mathbb{Z}_p) \) and \( H^*(B; \mathbb{Z}_p) \) are of finite type and \( p \) is an odd prime, then

\[ \text{Jet}^*(F; \mathbb{Z}_p) \cong \text{Tor}_{H^*(B; \mathbb{Z}_p)}(H^*(E; \mathbb{Z}_p)) \]

as an algebra over \( \mathbb{Z}_p \). (A similar result holds over the rationals \( \mathbb{Q} \).)

The result for \( p = 2 \) is more complicated to state and is treated in Theorem 3.

In fact, as we shall see, we can compute the indicated torsion product simply from a knowledge of the cohomology map

\[ \pi^*: H^*(B; \mathbb{Z}_p) \to H^*(E; \mathbb{Z}_p). \]

Results and techniques similar to these have been used in [8] to compute the \( \mathbb{Z}_p \)-cohomology of stable two stage Postnikov systems.

This announcement serves as an introduction to the joint work of J. C. Moore and the author that will appear elsewhere.

1. **Algebra.** Throughout this section \( k \) will denote a fixed field and \( \otimes \) will mean \( \otimes_k \). We shall assume that the reader is familiar with the material covered in the homological algebra section of [1]. All modules are assumed of finite type. All algebras will be assumed graded augmented and connected.

**DEFINITION.** If \( \Gamma \) is a Hopf algebra over \( k \), an ideal \( I \subset \Gamma \) is called
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a Hopf ideal, iff
\[ \nabla(I) \subset \Gamma \otimes I + I \otimes \Gamma \]
where \( \nabla: \Gamma \to \Gamma \otimes \Gamma \) is the coproduct in \( \Gamma \).

**Proposition 1.** If \( \Gamma \) is a commutative and cocommutative Hopf algebra over \( k \), \( I \subset \Gamma \) is a Hopf ideal, then there exists a unique sub-Hopf algebra \( \Lambda \subset \Gamma \) such that \( I = \Lambda \circ \Gamma \). (\( \Lambda = \ker \Delta \to k \).)

**Proof.** Let \( \Omega = \Gamma / I \). Then \( \Omega \) is a Hopf algebra in a natural way and the natural map
\[ \nu: \Gamma \to \Omega \]
is an epimorphism of Hopf algebras. Passing to duals we obtain a monomorphism of Hopf algebras
\[ \nu^*: \Omega^* \to \Gamma^*. \]
Since \( \Gamma^* \) is commutative we can set \( A = \Gamma^*/\Omega^* \). Passing to duals again and identifying \( \Gamma \) with its double dual we obtain \( A^* \subset \Gamma^* \). If we set \( \Lambda = A^* \) it is straightforward to verify that \( \Lambda \) has the required properties (see for example Proposition 4.4 of [7]).

**Notation.** Let \( \Gamma, \ A \) be commutative and cocommutative Hopf algebras over \( k \), \( \phi: \Gamma \to A \) a map of Hopf algebras. Then \( \ker \phi \subset \Gamma \) is a Hopf ideal. Hence by Proposition 1 there is a sub-Hopf algebra \( \Lambda \) of \( \Gamma \) with \( \Lambda \circ \Gamma = \ker \phi \). We will adopt the notation subker \( \phi \) for \( \Lambda \).

**Proposition 2.** Suppose that \( \Gamma \) is a commutative, cocommutative Hopf algebra over \( k \), \( A \) is a Hopf algebra over \( k \) and \( \phi: \Gamma \to A \) is a map of Hopf algebras, then
\[ \text{Tor}_r(k, A) \cong A/\phi \otimes \text{Tor}_{\text{subker} \phi}(k, k) \]
as an algebra.

**Proof.** According to [7, 4.4] \( \Gamma \) is a free subker \( \phi \)-module. Hence by [2, Theorem 6.1 p. 349] we have a spectral sequence \( \{ E_r, d_r \} \) such that
\[ E_r \Rightarrow \text{Tor}_r(k, A), \]
and if \( \Omega = \Gamma/\phi \)
\[ E_2 = \text{Tor}_0(\text{Tor}_{\text{subker} \phi}(k, k), A). \]
But \( \Omega \subset A \) is a sub-Hopf algebra, hence by [7, 4.4] again, \( A \) is a free \( \Omega \)-module. Therefore the edge homomorphism of the spectral sequence provides an isomorphism
Finally as in [1, §2.3] one can show that \( \text{Tor}_{\text{subker}} \phi(k, k) \) is a trivial \( \Omega \)-module, and hence
\[
A \otimes Q \text{Tor}_{\text{subker}} \phi(k, k) \cong A/\Omega \otimes \text{Tor}_{\text{subker}} \phi(k, k)
\]
and the result follows. □

Notation. We shall adopt the notation \( P[x_1, \ldots, x_n, \ldots] \) for a graded polynomial algebra over \( k \) on generators \( x_1, \ldots \) of degree \( \deg x_1, \ldots \).

Similarly \( E[y_1, \ldots] \) will denote a graded exterior algebra on generators \( y_1, \ldots \).

We note that if the characteristic of \( k \) is not 2 then \( \deg x_1, \ldots \) are all even.

We are now ready to make our main calculation. We therefore make the following assumptions:

(1) \( k = \mathbb{Z}_p \), \( p \) any prime or \( k = \mathbb{Q} \) the rational numbers.

(2) \( \Gamma \) is a Hopf algebra over \( k \)
   (a) As an algebra \( \Gamma \cong P[x_1, \ldots] \).
   (b) As a coalgebra \( \Gamma \) is commutative.

(3) \( A \) is a Hopf algebra over \( k \) and \( \phi: \Gamma \to A \) is a map of Hopf algebras.

Main Calculation. Under the above conditions
\[
\text{Tor}_r(A, k) \cong A/\phi \otimes E[u_1, \ldots]
\]
where
\[
u_i \subseteq \text{Tor}_r^1(A, k) \quad i = 1, \ldots.
\]

Proof. By Proposition 2,
\[
\text{Tor}_r(A, k) \cong A/\phi \otimes \text{Tor}_{\text{subker}} \phi(k, k).
\]

By construction subker \( \phi \subset \Gamma \) is a sub-Hopf algebra. By Borel's structure theorem for Hopf algebras over \( k \) [7, 7.11] subker \( \phi \cong P[v_1, \ldots] \) and the result now follows by the graded version of [6, Theorem 2.2, p. 205]. □

2. Multiplicative fibre maps. Suppose that \( F \to E \to B \) is a Serre fibre space, \( B \) simply connected and all cohomology in sight is of finite type.

Theorem (Eilenberg-Moore [3]). There exists a second quadrant spectral sequence \( \{ E_r, d_r \} \) such that

(1) \( E_r \to H^*(F; k) \).
\( E_2^{pq} = \text{Tor}_{H_{B}^n(B; \kappa)}(k, H^*(E; \kappa)) \), \( p \leq 0 \).

(3) \( E_r \) is in a natural way an algebra and \( d_r \) is a derivation of degree \((r, 1-r)\).

**Theorem 3.** Suppose that

\[ F \rightarrow E \rightarrow B. \]

is a multiplicative fibre map over the simply connected base space \( B \). In addition assume that \( k = \mathbb{Z}_p \), \( p \) any prime or \( k = \mathbb{Q} \) and

(1) \( H^*(B; \kappa) = \mathbb{P} \left[ x_1, \ldots \right] \).

(2) \( H_*(B; \kappa) \) is commutative.

Let \( \{ E_r, d_r \} \) denote the Eilenberg-Moore spectral of

\[ F \rightarrow E \rightarrow B. \]

Then

(1) \( E_2 \cong H^*(E; \kappa) \big/ \pi^* \otimes E[u_1, \ldots] \) as an algebra, where \( u_i \in E_2^{-1} \).

(2) \( E_2 = E_0 = E^0_\kappa H^*(F; \kappa) \).

**Proof.** (1) follows directly from the main calculation of the first section. To see (2) observe that

\[ d_r( E_2^{-p,*} ) \subseteq E_3^{-p+r,*} = 0 \] if \( p = 0, 1 \) and \( r \geq 2 \).

Hence \( d_r \) vanishes on the algebra generators of \( E_2 \) and since it is a derivation we must have \( d_r = 0, r \geq 2 \). □

**Corollary 4.** If \( k = \mathbb{Z}_p \), \( p \) an odd prime, or \( k = \mathbb{Q} \), then under the hypotheses of Theorem 3 there is an isomorphism of algebras

\[ H^*(F; \kappa) \cong \text{Tor}_{H^*(B; \kappa)}(k, H^*(E; \kappa)). \]

**Proof.** One merely notes that for a suitable filtration

\[ E^0 H^*(F, \kappa) \cong H^*(E; \kappa) \big/ \pi^* \otimes E[u_1, \ldots] \]

\[ \cong \text{Tor}_{H^*(B; \kappa)}(k, H^*(E; \kappa)) \]

and that \( E[u_1, \ldots] \) is a free commutative algebra. The result now follows by standard arguments. □

**Remark 1.** Theorem 3 can be used to calculate the \( \mathbb{Z}_2 \)-cohomology of stable two-stage Postnikov systems. (See [4], [5], [8].) It can also be used to simplify somewhat the calculations of [9]. From these calculations one can obtain the \( \mathbb{Z}_2 \) cohomology of the stages in the Postnikov tower of \( \text{SO} \).
Remark 2. It may be of interest, when more results are in, to apply Corollary 4 to the fibration

$$PL \to F \to F/PL.$$ 

References

9. R. Stong, *Determination of $H^*(BO(k, · · · , \infty); Z_2)$ and $H^*(BU(k, · · · , \infty); Z_2)$*, Trans. Amer. Math. Soc. 107 (1963), 526–544.

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