THE LOCAL RING OF THE GENUS THREE MODULUS SPACE AT KLEIN'S 168 SURFACE

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Communicated by E. Calabi, November 28, 1966

1. **Introduction.** In [6], as a synthesis of earlier papers of mine, I give, in the form of a set of prescriptions for local coordinates, a description of M^g, the space of conformal equivalence classes of compact Riemann surfaces of genus g, as a complex space. Particular interest attaches to those points (surface classes) of M^g representing surfaces admitting conformal self-maps (automorphisms) because, outside of certain cases for $g=1, 2, 3$ (over and above the elliptic and hyperelliptic involutions for $g=1, 2$), these points are singular (nonuniformizable) points in the structure. In particular for $g\geqq 2$, where one needs $3g-3$ complex parameters to describe M^g near a generic point, one needs $3g-3+\rho$, $\rho>0$, near one of the points in question.[2] According to Prescription III ([6, p. 17]) the problem reduces to finding an irreducible basis for the homogeneous nonconstant, polynomial invariants of a finite group of linear transformations in $3g-3$ variables, namely, the hermitian adjoint of the group induced on the quadratic differentials of a representative surface of the point in question by the conformal automorphism group of that surface.

For a finite nonabelian linear group, while there is an algorithm for computing some basis for the invariants (cf. Prescription III), there is notoriously no known algorithm for computing an *irreducible* basis, i.e., for discarding the superfluous ones. Accordingly I felt it of interest to illustrate the whole phenomenon by a nontrivial example. To anyone who has worked on the subject the one that immediately comes to mind is Klein's surface of genus three admitting as automorphism group a representation of the simple group of order 168 ([1], [3]). This example commends itself in that it is of "maximum complexity" in the sense that it admits its full quota according to Hurwitz [2] of $84(g-1)=168$, $(g=3)$ automorphisms (on the subject of such surfaces see the interesting papers [4], [5]). It develops, *vide infra*, that eleven invariants, i.e., $11=3\cdot 3-3+5=6+5$, so $\rho=5$, are needed to generate the local ring of M^3 at Klein's surface class.[2]

[1] Research partially sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Grant No. AF-AFOSR-1077-66.

[2] With ρ relations on "syzygies," of course.

2. **The quadratic differentials on Klein's surface.** Klein's surface S is the compact Riemann surface of genus three obtained by identifying the upper half-plane under the principal congruence subgroup of level seven of the inhomogeneous modular group with appropriate conventions at cusps and vertices. The factor group G acts on S as its group of conformal automorphisms representing faithfully thereby the unique simple group of order 168.

LEMMA 1 ([**3**, p. 444]). *There is a basis z_1, z_2, z_3 of the abelian differentials of first kind on S on which G induces a representation $R_3(G)$ generated by*

$$\begin{aligned} T: z_1' &= \epsilon z_1, \qquad z_2' = \epsilon^4 z_2, \qquad z_3' = \epsilon^2 z_3 \\ U: z_1' &= az_1 + bz_2 + cz_3 \\ z_2' &= bz_1 + cz_2 + az_3 \\ z_3' &= cz_1 + az_2 + bz_3, \end{aligned} \tag{1}$$

where $\epsilon = e^{2\pi i/7}$, $\sqrt{-7} = \epsilon + \epsilon^2 + \epsilon^4 - \epsilon^3 - \epsilon^5 - \epsilon^6$, and $a = (\epsilon^5 - \epsilon^2)/\sqrt{-7}$, $b = (\epsilon^3 - \epsilon^4)/\sqrt{-7}$, $c = (\epsilon^6 - \epsilon)/\sqrt{-7}$ are all real. T has order 7, and U has order 2.

LEMMA 2. *As a basis for the quadratic differentials on S one can choose*

$$\begin{gathered} \xi_1 = z_1^2, \qquad \xi_2 = z_2^2, \qquad \xi_3 = z_2^2, \\ \xi_4 = \sqrt{2}\, z_1 z_2, \qquad \xi_5 = \sqrt{2}\, z_2 z_3, \qquad \xi_6 = \sqrt{2}\, z_1 z_3. \end{gathered} \tag{2}$$

The representation $R_6(G)$ induced on (2) by $R_3(G)$ is unitary.

PROOF. S is not hyperelliptic ([**3**, p. 437]), hence Noether's theorem ([**6**, Lemma 5]) implies that any six distinct quadratic products of the z's, in particular (2), span the quadratic differentials. By Lemma 1, $R_6(G)$ is generated by

$$\begin{aligned} T_2: \xi_1' &= \epsilon^2 \xi_1,\ \xi_2' = \epsilon \xi_2,\ \xi_3' = \epsilon^4 \xi_3,\ \xi_4' = \epsilon^5 \xi_4, \xi_5' = \epsilon^6 \xi_5, \xi_6' = \epsilon^3 \xi_6, \\ U_2: \xi_1' &= a^2 \xi_1 + b^2 \xi_2 + c^2 \xi_3 + \sqrt{2}\, ab \xi_4 + \sqrt{2}\, bc \xi_5 + \sqrt{2}\, ac \xi_6, \cdots, \\ \xi_4' &= \sqrt{2}\, ab \xi_1 + \sqrt{2}\, bc \xi_2 + \sqrt{2}\, ac \xi_3 + (ac + b^2) \xi_4 \\ &\quad + (ab + c^2) \xi_5 + (bc + a^2) \xi_6, \cdots, \end{aligned} \tag{3}$$

where the dots signify cyclic permutation of a, b, c. T_2 is clearly unitary. U_2 is real symmetric by inspection and of order 2 by Lemma 1, hence real orthogonal, *a fortiori* unitary.

LEMMA 3. *Define*

$$\begin{aligned} \gamma_k(\xi) &= \alpha(\epsilon^{-2k}\xi_1 + \epsilon^{-k}\xi_2 + \epsilon^{-4k}\xi_3) + \beta(\epsilon^{-5k}\xi_4 + \epsilon^{-6k}\xi_5 + \epsilon^{-3k}\xi_6), \\ \bar{\gamma}_k(\xi) &= \beta(\epsilon^{2k}\xi_1 + \epsilon^{k}\xi_2 + \epsilon^{4k}\xi_3) + \alpha(\epsilon^{5k}\xi_4 + \epsilon^{6k}\xi_5 + \epsilon^{3k}\xi_6), \end{aligned} \tag{4}$$

where $\alpha^2 = (-1+\sqrt{-7})/\sqrt{8} = (\epsilon+\epsilon^2+\epsilon^4)/\sqrt{2}$, $\beta=\bar{\alpha}$, $\beta^2 = (\epsilon^3+\epsilon^5+\epsilon^6)/\sqrt{2}$, $\alpha\bar{\alpha}=\beta\bar{\beta}=1$, *and* $k=0, \cdots, 6$. *One has* (i) $R_6(G)$ *induces on the* $\gamma_k(\xi)$ *and the* $\bar{\gamma}_k$ *respectively the degree seven permutation representation* $R_7(G)$ *of* G *and its inverse* $R_7^{-1}(G)$ *and* (ii)

$$\Sigma\gamma_k(\xi) = \Sigma\bar{\gamma}_k(\xi) = 0, \tag{5}$$

$$\begin{aligned} \gamma_{1-k}(\xi) &= (1/\sqrt{2})(\bar{\gamma}_k(\xi)+\bar{\gamma}_{k+1}(\xi)+\bar{\gamma}_{k+3}(\xi)), \\ \bar{\gamma}_{1-k}(\xi) &= (1/\sqrt{2})(\gamma_k(\xi)+\gamma_{k+1}(\xi)+\gamma_{k+3}(\xi)), \end{aligned} \tag{6}$$

where k *is computed* mod 7, *and*

$$\xi_1 = (1/7\alpha)\Sigma\epsilon^{2k}\gamma_k(\xi), \cdots. \tag{7}$$

PROOF. (i) and (ii) follow from the substitution of (2) in the formulae in [**1**, p. 519] and [**1**, II, p. 459 and p. 501] and the corresponding facts noted there. However, once aware of (5), (6), (7) one easily verifies them directly. As for (i) one sees immediately that T_2 generates the cyclic permutation (6543210) of the indices of the $\gamma_k(\xi)$ and the inverse permutation of the $\bar{\gamma}_k(\xi)$. Calculation reveals that U_2 induces the (self-inverse) permutation (12)(36) on both $\gamma_k(\xi)$ and $\bar{\gamma}_k(\xi)$. These permutations generate the representations of (i). (7) implies their faithfulness.

3. Invariants of $R_6(G)$ and main theorem.

THEOREM A (GORDAN [**1**]). *Let* x_k, $\bar{x}_k$, $k=0, \cdots, 6$ *be two sets of variables which satisfy conditions* (i) *and* (ii) *of Lemma* 3 *when formally substituted for* $\gamma_k(\xi)$, $\bar{\gamma}_k(\xi)$ *respectively. Then any* (*nonconstant, homogeneous*) *polynomial in* $x_0, \cdots, x_6$ *over the ring of integers which is invariant under* $R_7(G)$ *is a polynomial over* $R(\sqrt{2})$ (R *is the rationals*) *in the power sums* S_i, $\bar{S}_i$, $i=1, \cdots, 7$ *of the* x*'s and* $\bar{x}$*'s* (*separately*). *By assumption* $S_1=\bar{S}_1=0$. *One easily verifies* $S_2=\bar{S}_2$. *The basis of eleven invariants* S_2, S_j, $\bar{S}_j$, $j=3, \cdots, 7$ *is irreducible, i.e., none is a polynomial* (*over* **C**) *in the others.*

Gordan also computes explicitly a set of five syzygies but does not prove they are a basis for all syzygies.

Define, for a set of variables $\lambda_1, \cdots, \lambda_6$ and positive integral j

$$S_j(\lambda) = \Sigma(\gamma_k(\lambda))^j, \qquad \bar{S}_j = \Sigma(\bar{\gamma}_k(\lambda))^j. \tag{8}$$

THEOREM 1. *Let* $t\in T^3$ (*Teichmueller space of genus three*) *lie over* $[S]\in M^3$ (*modulus space of genus three*), *where* S *is Klein's surface. One can introduce coordinates in* T^3 *near* t *such that the local ring of* M^3 *at* $[S]$ *is generated by the irreducible basis of eleven distinct homogeneous polynomials obtained by setting* $j=2, \cdots, 7$ *in* (8).

PROOF. Introducing $\lambda_1, \cdots, \lambda_6$ by Prescription III, using (2) as a basis for $A(S)$ ([6, p. 17—in Proposition 8 the reference should be to Prescription I, not II]), I have to find an irreducible basis for the nonconstant homogeneous polynomials in the λ's invariant under the hermitian adjoints of the matrices of $R_6(G)$. But by Lemma 2 and the group property this will be identical with a basis for the invariants of $R_6(G)$. Lemma 3 and Theorem A with

$$x_k = \gamma_k(\lambda), \qquad \bar{x}_k = \bar{\gamma}_k(\lambda), \qquad k = 0, \cdots, 6 \tag{9}$$

show that the invariants of Theorem 1 are a basis. The sticky point is that under the specialization (9) they might reduce. However, Gordan in [1, II, p. 461], says that even under the more severe specialization (9) and $\lambda_1\lambda_2 = 2\lambda_4^2$, etc., the worst that can happen is $\beta S_3(\lambda) = \alpha \bar{S}_3(\lambda)$. A computation shows that this does not happen without the additional specialization.

BIBLIOGRAPHY

1. P. Gordan, *Ueber Gleichungen siebenten Grades mit einer Gruppe von 168 Substitutionen*, Math. Ann. **20** (1882), 515–530; II, ibid. **25** (1885), 459–521.

2. A. Hurwitz, *Über algebraische Gebilde mit eindeutigen Transformationen in sich*, Mathematische Werke, Bd. I, Birkhäuser, Basel, 1932, pp. 392–430.

3. F. Klein, *Ueber die Transformation siebenter Ordnung der elliptischen Funktionen*, Math. Ann. **14** (1879), 428–471.

4. A. M. Macbeath, *On a theorem of Hurwitz*. Proc. Glasgow Math. Assoc. **5** (1961), 90–96.

5. ———, *On a curve of genus* 7, Proc. London Math. Soc. **15** (1965), 527–542.

6. H. E. Rauch, *A transcendental view of the space of algebraic Riemann surfaces*, Bull. Amer. Math. Soc. **71** (1965), 1–39.

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