INTRINSIC METRICS ON COMPLEX MANIFOLDS

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1. Definition of intrinsic pseudometric. Let M be a (connected) complex manifold. We shall define a pseudometric d on M in a natural manner so that it depends only on the complex structure of M and nothing else.

Let D be the open unit disk in the complex plane and ρ the distance on D defined by the Poincaré-Bergman metric of D. Given two points p and q of M, choose the following objects:

(1) points $p=p_0, p_1, \cdots, p_{k-1}, p_k=q$ of M and

(2) points $a_1, \cdots, a_k, b_1, \cdots, b_k$ of D and holomorphic mappings $f_1, \cdots, f_k$ of D into M such that $f_i(a_i)=p_{i-1}$ and $f_i(b_i)=p_i$ for $i=1, \cdots, k$.

For each choice of points and mappings satisfying (1) and (2), consider the number $\rho(a_1, b_1)+\cdots+\rho(a_k, b_k)$. Let $d(p, q)$ be the infimum of the numbers obtained in this manner for all possible choices. It is easy to verify that d is a pseudometric on M in the sense that

$$d(p, q) \geqq 0, \quad d(p, q) = d(q, p), \quad d(p, q) + d(q, r) \geqq d(p, r)$$

for $p, q, r \in M$. The following two propositions are immediate from the definition of d.

PROPOSITION 1. *Let M and N be two complex manifolds and d_M and d_N the intrinsic pseudometrics of M and N. Then every holomorphic mapping $f: M \to N$ is distance-decreasing in the sense that*

$$d_M(p, q) \geqq d_N(f(p), f(q)) \quad \text{for } p, q \in M.$$

In particular, every holomorphic transformation of M is distance-preserving with respect to d_M.

PROPOSITION 2. *For the complex Euclidean space C^n, the pseudometric d is trivial, i.e., $d(p, q)=0$ for all $p, q \in C^n$.*

The following proposition follows from the Schwarz-Pick lemma.

PROPOSITION 3. *For the unit disk D, the pseudometric d coincides with the distance ρ defined by the Poincaré-Bergman metric.*

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The following theorem follows from a generalized Schwarz-Pick lemma (see [3] and [4]).[2]

THEOREM 4. *Let M be a hermitian manifold whose holomorphic sectional curvature is negative and bounded away from zero. Then its pseudometric d is a metric, i.e., $d(p, q)=0$ implies $p=q$.*

Theorem 4 applies to all bounded domains of C^n as well as to all Riemann surfaces of hyperbolic type.

2. **Relationship with Carathéodory metric.** Following Carathéodory [2] we define another pseudometric d' on a complex manifold M. Given two points p and q of M, let $d'(p, q)$ be the supremum of $\rho(f(p), f(q))$ for all holomorphic mappings f of M into the unit disk D.

It is easy to see that Propositions 1, 2 and 3 above hold also for the Carathéodory pseudometric d'. A necessary and sufficient condition for d' to be a metric on M is that there are sufficiently many bounded holomorphic functions on M so that they separate the points of M.

The following proposition is immediate from Propositions 1 and 3 and shows that whenever the Carathéodory pseudometric d' is a metric, our pseudometric d is also a metric.

PROPOSITION 5. *For any complex manifold M, d is greater than or equal to d', i.e.,*

$$d(p, q) \geqq d'(p, q) \quad \text{for } p, q \in M.$$

3. **Applications.** The following theorem which follows from Propositions 1 and 2 and Theorem 4 may be considered as a generalization of Picard Theorem which states that an entire function with more than one finite lacunary value reduces to a constant function.

THEOREM 6. *Let M be an n-dimensional complex manifold which admits a hermitian metric whose holomorphic sectional curvature is negative and bounded away from zero. Then every holomorphic mapping f of C^m into M is a constant map.*

COROLLARY. *The complex Euclidean space C^m does not admit a hermitian metric whose holomorphic sectional curvature is negative and bounded away from zero.*

The condition "bounded away from zero" is essential. In fact, C

[2] The results proved for Kähler manifolds in [4] hold for hermitian manifolds (with respect to the hermitian connection in the sense of Chern). Proofs there remain valid in the hermitian case.

(and hence C^m) admits a complete Kähler metric of negative holomorphic sectional curvature, e.g.,

$$(1 + z\bar{z})\, dzd\bar{z}.$$

THEOREM 7. *Let M be a hermitian manifold whose holomorphic sectional curvature is negative and bounded away from zero. Then the group of holomorphic transformations of M is a Lie group with compact isotropy subgroups (with respect to the compact-open topology).*

In fact, the group in question is a closed subgroup of the group of isometries of M with respect to the intrinsic metric d introduced above. By a classical theorem of Van Dantzig and van der Waerden (see Theorem 4.7 and Corollary 4.8 of Chapter I in [5]) the group of isometries of a locally compact metric space is locally compact and its isotropy subgroups are all compact with respect to the compact-open topology. Theorem 7 follows now from a well-known theorem of Bochner-Montgomery [1].

COROLLARY 8. *If, in Theorem 7, M is moreover compact, then the group of holomorphic transformations of M is a finite group.*

The results in this section have been obtained by Wu [6] using the notion of normal families.

BIBLIOGRAPHY

1. S. Bochner and D. Montgomery, *Locally compact groups of differentiable transformations*, Ann. of Math. **47** (1946), 639–653.

2. C. Carathéodory, *Über das Schwarzsche Lemma bei analytischen Funktionen von zwei komplexen Veränderlichen*, Math. Ann. **97** (1926), 76–98.

3. H. Grauert and H. Reckziegel, *Hermitesche Metriken und normale Familien holomorphic Abbildungen*, Math. Z. 89 (1965), 108–125.

4. S. Kobayashi, *Distance, holomorphic mappings and the Schwarz's lemma*, J. Math. Soc. Japan (to appear).

5. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Interscience Tracts No. 15, Interscience, New York, 1963.

6. H. H. Wu, *Normal families of holomorphic mappings and the theorem of Bloch in several complex variables* (to appear).

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