# ON THE CLOSURE OF CERTAIN BANACH SPACES OF FUNCTIONS OF SEVERAL VARIABLES

BY DAVID A. SPRECHER[1]

Communicated by L. Nirenberg, January 13, 1967

**1. Statement of the main results.** Our primary goal in this note is to establish the following proposition.

THEOREM 1. *Consider the Banach space* $\mathcal{C}_n$ *of continuous real-valued functions* $f\colon E_n \to R$, $E_n$ *standing for the unit cube in n-dimensional Euclidean space. If* $\phi$ *and* $\psi$ *are any fixed functions of* $\mathcal{C}_n$ *with connected level-sets intersecting pairwise in connected sets, then the subspace of superpositions* $a \circ \phi + b \circ \psi$ *is closed in* $\mathcal{C}_n$ *under the uniform norm.*

Mark by $\mathcal{B}_n$ the indicated space of superpositions. To prove the stated theorem, it suffices to verify

THEOREM 2. *Every function of* $\mathcal{C}_n$ *has a best uniform approximation in* $\mathcal{B}_n$.

Distinguish one of the fixed functions, say, $\psi$; denote its level sets by $l_\psi(t)$,

$$l_\psi(t) = \{p \in E_n : \psi(p) = t\};$$

designate by $L_\psi$ the aggregate of level sets $l_\psi = l_\psi(t)$. Finally, set for each $f \in \mathcal{C}_n$

$$\omega(f \mid l_\psi) = \max_{p \in l} f(p) - \min_{p \in l_\psi} f(p),$$

$$\omega(f \mid \psi) = \max_{l_\psi \in L_\psi} \omega(f \mid l_\psi),$$

$$\mu(f) = \inf_{\mathcal{B}_n} \|f - a \circ \phi - b \circ \psi\|,$$

(the properties of the functional $\omega$ and related topics are investigated in [1] and [2]. Theorem 2 is proved by means of the five lemmas now formulated.

LEMMA 1. *For each* $f \in \mathcal{C}_n$,

$$\mu(f) = \tfrac{1}{2} \inf_{a \in \mathcal{C}} \omega(f - a \circ \phi \mid \psi),$$

[1] This research was supported in part by the National Science Foundation under Grant No. GP-4165.

*where* $\mathcal{C}$ *stands for the appropriate space of continuous functions of a single variable.*

LEMMA 2. *The function* $\alpha \circ \phi + \beta \circ \psi$ *of* $\mathcal{B}_n$ *is a best approximation to* $f \in \mathcal{C}_n$ *if* $\alpha$ *minimizes the functional* $\omega$,

$$\omega(f - \alpha \circ \phi \mid \psi) = \inf_{a \in \mathcal{C}} \omega(f - a \circ \phi \mid \psi),$$

and $\beta \circ \psi$ is defined for each $l_\psi$ as

$$\beta \circ \psi = \tfrac{1}{2}[\max_{l\psi}(f - \alpha \circ \phi) + \min_{l\psi}(f - \alpha \circ \phi).$$

Since $\omega$ measures the oscillation of $f$ on the partition $L_\psi$ (of $E_n$), we agree to call $\alpha$ a best $\omega$-approximation of $f$ (see also [3]).

LEMMA 3. *Given a function* $f \in \mathcal{C}_n$, *consider the family* $\mathfrak{f}$ *whose members are defined for each admitted level set* $l_\psi(t)$ *to be the restriction of* $f$ *to it*:

$$\mathfrak{f} = \{f_t : f_t = f \mid l_\psi(t)\};$$

*let* $\mathfrak{f}_\epsilon \subset \mathfrak{f}$ *be an* $\epsilon$*-net of the family* $\mathfrak{f}$, *then*

$$\inf_{a \in \mathcal{C}} \omega(\mathfrak{f}_\epsilon - a \circ \phi \mid \psi) \leqq \mu(f) \leqq \inf_{a \in \mathcal{C}} \omega(\mathfrak{f}_\epsilon - a \circ \phi \mid \psi) + \epsilon,$$

*where*

$$\omega(\mathfrak{f}_\epsilon - a \circ \phi \mid \psi) = \max_{f_t \in \mathfrak{f}_\epsilon} \omega(f_t - a \circ \phi \mid l_\psi(t)).$$

LEMMA 4. *Each finite family* $\mathfrak{f}_\epsilon$ *has a best* $\omega$*-approximation.*

LEMMA 5. *Corresponding to the members* $\epsilon_k > 0$ *of the convergent series* $\sum \epsilon_k$, *let* $\{\mathfrak{f}_k\}$ *be an ascending chain of* $\epsilon_k$*-nets of* $\mathfrak{f}$:

$$\mathfrak{f}_1 \subset \mathfrak{f}_2 \subset \cdots \subset \mathfrak{f}_k \subset \cdots \subset \mathfrak{f}.$$

*There is a uniformly convergent sequence of best* $\omega$*-approximations* $\alpha_k$ *to* $\mathfrak{f}_k$ *whose limit is a best* $\omega$*-approximation to the function* $f$ *attached to* $\mathfrak{f}$.

Theorem 1 was arrived at through an examination of the constructions in [2]. Specifically, the first three lemmas above correspond to the first three theorems in [2]; Lemma 5 is the counterpart of [2, Theorem 4]. The proofs of the preceding lemmas require certain modifications of the arguments in [2], based on these facts:

Given $f$, let $\mathfrak{f}_\epsilon = \{f_1, \cdots, f_m\}$, $f_j = f \mid l_\psi(t_j)$, be a fixed $\epsilon$-net of $\mathfrak{f}$; let $\mathfrak{f}_{\epsilon k}$, $1 \leqq k \leqq m$, consist of the first $k$ elements of $\mathfrak{f}_\epsilon$.

(i) If $\alpha_k$ is a best $\omega$-approximation of $\mathfrak{f}_{\epsilon k}$, then so is $\alpha_k + c$ for each constant $c$.

(ii) If $\alpha_{k+1}$ is a best $\omega$-approximation of $\mathfrak{f}_{\epsilon k+1}$, then there is a constant $c$ such that for each point $p \in l_\psi(t_{k+1})$, $\operatorname{sgn}[f_{k+1}-\alpha_{k+1} \circ \phi - c] = \operatorname{sgn}[\alpha_{k+1} \circ \phi + c - \alpha_k \circ \phi]$, unless $f_{k+1} = \alpha_{k+1} \circ \phi + c$ or $\alpha_k \circ \phi = \alpha_{k+1} \circ \phi + c$.

2. **On the choice of $\phi$ and $\psi$.** The following propositions are conveniently formulated with this notation:

*Consider two functions, $\psi_1$ and $\psi_2$, of $\mathcal{C}_n$: if $L_{\psi_1}$ is a refinement of $L_{\psi_2}$, when these are regarded as partitions of $E_n$, then we write $\psi_2 \succ \psi_1$ (see* [1]). *Designating now best approximations of $f$ in $\mathcal{B}_n$ relative to the fixed functions $\phi$ and $\psi$ by $\mu(f; \phi, \psi)$, we state*

THEOREM 3. *If $\phi_2 \succ \phi_1$ and $\psi_2 \succ \psi_1$, then*

$$\mu(f; \phi_2, \psi_2) \geqq \mu(f; \phi_1, \psi_1)$$

*for each $f \in \mathcal{C}_n$.*

THEOREM 4. *If $\phi \succ \psi$, then*

$$\mu(f; \phi, \psi) = \tfrac{1}{2}\omega(f \mid \psi).$$

THEOREM 5. *For each $p \in E_n$ and $i = 1, 2$, let $U_{ip}$ designate the intersection of level sets $l_{\phi_i}$ and $l_{\psi_i}$ containing $p$ (that is, we remove now the restriction imposed in the fixed functions in Theorem 1). If $U_{2p} \supset U_{1p}$ for each $p \in E_n$, then the conclusion of Theorem 3 remains valid for each $f \in \mathcal{C}_n$.*

Theorem 3 is a generalization of [1, Corollary 3.2]. To prove it, consider first the case when $\phi_1 = \phi_2$. Given $f$, let $\{a_k\}$ be a sequence of members of $\mathcal{C}$ such that

$$\lim_{k \to \infty} \omega(f - a_k \circ \phi_2 \mid \psi_2) = \mu(f; \phi_2, \psi_2):$$

clearly, we are guaranteed the existence of such sequences. Owing to [1, Corollary 2.1], we have the inequality

$$\omega(f - a_k \circ \phi_2 \mid \psi_2) \geqq \omega(f - a_k \circ \phi_2 \mid \psi_1)$$

for each fixed $k$, and consequently this remains true as we let $k \to \infty$; while we are not assured the convergence of the sequence whose members are $\omega(f - a_k \circ \phi_2 \mid \psi_1)$, we can state that

$$\mu(f; \phi_2, \psi_2) \geqq \limsup_{k \to \infty} \omega(f - a_k \circ \phi_2 \mid \psi_1).$$

If it happens that the right side of this inequality equals $\mu(f; \phi_2, \psi_1)$ then we have established this case; otherwise, we consider a sequence, $\{c_k\}$, of functions $c_k \in \mathcal{C}$, subject to the specification

$$\lim_{k\to\infty} \omega(f - c_k \circ \phi_2 \mid \psi_1) = \mu(f; \phi_2, \psi_1).$$

A comparison of the last two inequalities reveals at once that

$$\mu(f; \phi_2, \psi_2) \geqq \mu(f; \phi_2, \psi_1),$$

thereby disposing of the case in question.

The proof of the theorem is completed by applying Lemma 1 above to the last inequality, and using the fact that

$$\mu(f, \phi, \psi) = \inf_{a\in\mathcal{C}} \inf_{b\in\mathcal{C}} \|f - a\circ\phi - b\circ\psi\| = \inf_{b\in\mathcal{C}} \inf_{a\in\mathcal{C}} \|f - a\circ\phi - b\circ\psi\|.$$

Theorem 4 follows at once from the fact that under these circumstances $\phi = g \circ \psi$ for some function $g \in \mathcal{C}$ (see [1]).

Theorem 5 is verified with the basic inequality $\omega(f\mid S) \geqq \omega(f\mid T)$ whenever $S \supset T$ and the fact that if $l_\phi \cap l_\psi = S$, then

$$\mu(f\mid S; \phi, \psi) = \tfrac{1}{2}\omega(f\mid S).$$

In particular, this theorem implies that in order to optimize our best approximations, all members of $L_\phi$ should intersect those of $L_\psi$ in at most one point.

In connection with these results we pose this problem:

It is well known that certain partitions of $E_n$ cannot qualify for membership in the family of level sets of a continuous function. Thus, for example, when $L_f = E_n$, then $f$ is at most continuous almost everywhere. We ask, therefore, for necessary and sufficient conditions for a partition $P$ of $E_n$ to admit a function $f \in \mathcal{C}_n$, such that $P = L_f$; similarly, under what circumstances can the function so associated with $P$ be continuous merely almost everywhere?

## Bibliography

1. D. A. Sprecher, *On best approximations in several variables*, J. Reine Angew. Math (to appear).

2. ———, *On best approximations of functions of two variables*, (to appear).

University of California, Santa Barbara