SOME REMARKS ON THE EXTENDED DOMAIN OF FOURIER TRANSFORM

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1. Preliminaries. We shall state in this section definitions and results concerning general integral transformations, which will be needed in the second part of the paper. The proofs of these results can be found in [1].

Let (X, μ), (Y, ν) be two σ-finite measure spaces, ϕ and ψ be measurable, positive almost everywhere functions on X and Y, such that $\int_X \phi(x)d\mu(x) = \int_Y \psi(y)d\nu(y) = 1$ and denote by ρ_X and ρ_Y the translation invariant metrics defined for measurable, finite a.e. complex valued functions on X and Y by the formulas

$$\rho_X(u) = \int_X \frac{|u(x)|}{1 + |u(x)|}\phi(x)d\mu(x), \tag{1}$$
$$\rho_Y(v) = \int_Y \frac{|v(y)|}{1 + |v(y)|}\psi(y)d\nu(y).$$

The spaces $\mathfrak{M}$ and $\mathfrak{N}$ of all measurable finite a.e. functions on X and Y respectively, when provided with metrics ρ_X and ρ_Y become complete linear metric spaces. The topologies induced by these metrics are equivalent to topologies of convergence in measure on all subsets of finite measure.

If $K(x, y)$ is a measurable, complex valued function defined on $X \times Y$, then the proper domain of the corresponding integral transformation K is defined by

$$\mathfrak{D}_K = \left\{u \in \mathfrak{M}: (|K||u|)(y) = \int_X |K(x, y)||u(x)|\, d\mu(x) < \infty \text{ a.e. } \nu\right\} \tag{2}$$

and for $u \in \mathfrak{D}_K$ the integral transformation $K: \mathfrak{M} \to \mathfrak{N}$ is given by

$$(Ku)(y) = \int_X K(x, y)u(x)d\mu(x). \tag{3}$$

K is nonsingular if there exists a function $u \in \mathfrak{D}_K$ such that $u > 0$ a.e. From now on we suppose that K is nonsingular. If A is a linear metric space continuously contained in $\mathfrak{M}$ then K is called A-semiregular (A-s.r.) if (i) $\mathfrak{D}_K \cap A$ is dense in A, (ii) the mapping $K: A \cap \mathfrak{D}_K \to \mathfrak{N}$ is continuous if $A \cap \mathfrak{D}_K$ is provided with the metric of A. If K is A-s.r. then it can be extended to the whole of A as a continuous transformation which we denote by K_A. A is an FL subspace of $\mathfrak{M}$ if A is a complete linear metric space continuously contained in $\mathfrak{M}$ and $u \in A$, $v \in \mathfrak{M}$, $|v(x)| \leqq |u(x)|$ a.e. imply that $v \in A$.

Define for every $f \in \mathfrak{M}$

$$(4) \quad \tilde{\rho}_K(f) = \rho_X(f) + \sup\{\rho_Y(Kg) : g \in \mathfrak{D}_K,\ |g(x)| \leqq |f(x)| \text{ a.e.}\}.$$

PROPOSITION 1. (i) *$\tilde{\rho}_K$ is a complete metric on $\mathfrak{M}$*; (ii) *the closure $\tilde{\mathfrak{D}}_K$ of $\mathfrak{D}_K$ in $\mathfrak{M}$ with the metric $\tilde{\rho}_K$ is an* FL *subspace of $\mathfrak{M}$*; (iii) *the transformation $K: \mathfrak{D}_K \to \mathfrak{N}$ is continuous if $\mathfrak{D}_K$ is provided with the metric $\tilde{\rho}_K$.*

REMARK. $\mathfrak{M}$ provided with the metric $\tilde{\rho}_K$ is in general only a complete additive group and not a linear metric space.

(ii), (iii) of Proposition 1 imply that K is $\tilde{\mathfrak{D}}_K$ s.r. The corresponding extension $K_{\tilde{\mathfrak{D}}_K}$ is denoted by $\tilde{K}$. $\tilde{\mathfrak{D}}_K$ is referred to as the extended domain of K.

THEOREM 1. *If A is an* FL *subspace of $\mathfrak{M}$ such that K is A-*s.r. *then A is continuously contained in $\tilde{\mathfrak{D}}_K$ and K_A is the restriction of $\tilde{K}$ to A.*

In this paper we are interested in the special case when $X = Y = R^1$, $\mu = \nu$ is the Lebesgue measure and $K(x, y) = \mathfrak{F}(x, y) = (2\pi)^{-1/2}e^{-ixy}$ is the kernel corresponding to the Fourier transform. Clearly $\mathfrak{D}_{\mathfrak{F}} = L^1(R^1)$. As concerns $\tilde{\mathfrak{D}}_{\mathfrak{F}}$ the following proposition is known

PROPOSITION 2. *If $u \in \tilde{\mathfrak{D}}_{\mathfrak{F}}$ then $u \in L^1_{\text{loc}}(R^1)$ and for every $a > 0$ the quantity*

$$(5) \qquad \|u\|_a = \left[\sum_{n=-\infty}^{\infty} \left(\int_{na}^{(n+1)a} |u|\, dx\right)^2\right]^{1/2}$$

is finite.

Theorem 1 implies that $L^1(R^1) + L^2(R^1) \subset \tilde{\mathfrak{D}}_{\mathfrak{F}}$; the existence of functions satisfying the condition in Proposition 2, which are not in $L^1 + L^2$ follows from the following elementary remark.

REMARK. A nonnegative function u belongs to $L^1 + L^2$ iff $\min(u, 1) \in L^2$ and $u - \min(u, 1) \in L^1$.

The function $u = n$ if $n \leqq x \leqq n + 1/n^2$, $n = 1, 2, \cdots$, and $u = 0$ elsewhere is not in $L^1 + L^2$ and $\|u\|_1$ is finite. It follows from the Proposition 3 below that $\|u\|_a$ is finite for every $a > 0$.

We shall prove in the next section that the condition in Proposition 2 is actually necessary and sufficient in order that $u \in \tilde{\mathfrak{D}}_{\mathfrak{F}}$. In particular this disproves the conjecture raised in [1] that $\tilde{\mathfrak{D}}_{\mathfrak{F}} = L^1 + L^2$.

2. **The extended domain of Fourier transform.** Denote by $\mathcal{L}_a^{(2)}$ the set of all functions u in L^1_{loc} for which $\|u\|_a$ is finite and let $\mathcal{L}^{(2)} = \bigcap_{a>0} \mathcal{L}_a^{(2)}$. Clearly $\|\ \|_a$ is a norm on $\mathcal{L}_a^{(2)}$; it can easily be verified that $\mathcal{L}_a^{(2)}$ is the completion of L^1 with this norm. The following proposition shows that $\mathcal{L}_a^{(2)} = \mathcal{L}^{(2)}$ for all $a > 0$.

PROPOSITION 3. *The norms $\|\ \|_a$ are equivalent for $a > 0$. More precisely, if $0 < a < b$ and k is the smallest integer such that $b/k < a$ then*

$$(2k)^{-1/2}\|u\|_b \leqq \|u\|_a \leqq 2^{1/2}\|u\|_b \tag{6}$$

for all $u \in \mathcal{L}^{(2)}$.

PROOF. Denote by k_n the integers such that $k_n a < nb \leqq (k_n+1)a$, $n = 1, 2, \cdots$. The second estimate in (6) is obtained by adding the inequalities

$$\begin{aligned}
\sum_{l=k_n+1}^{k_{n+1}} \left(\int_{la}^{(l+1)a} |u|\, dx\right)^2
&= \sum_{l=k_n+1}^{k_{n+1}-1} \left(\int_{la}^{(l+1)a} |u|\, dx\right)^2 \\
&\quad + \left[\left(\int_{k_{n+1}a}^{(n+1)b} + \int_{(n+1)b}^{(k_{n+1}+1)a}\right) |u|\, dx\right]^2 \\
&\leqq 2\left[\sum_{l=k_n+1}^{k_{n+1}-1} \left(\int_{la}^{(l+1)a} |u|\, dx\right)^2\right. \\
&\quad \left. + \left(\int_{(k_{n+1}-1)a}^{(n+1)b} |u|\, dx\right)^2 + \left(\int_{(n+1)b}^{k_{n+1}a} |u|\, dx\right)^2\right].
\end{aligned}$$

Similarly adding the inequalities

$$\int_{nb}^{(n+1)b} |u|\, dx \leqq \sum_{l=k_n}^{k_{n+1}} \int_{la}^{(l+1)a} |u|\, dx$$

we get the first estimate in (6).

In the outline of the proof above we considered only the part of the sum in $\|\ \|_a$ and $\|\ \|_b$ corresponding to $n > 0$; the terms with negative indices n are taken care of by symmetry.

If $u \in \mathcal{L}^{(2)}$ and ϕ is a function of rapid decrease, $\phi \in \mathcal{S}$, then the integral $\int_{-\infty}^{\infty} u\phi\, dx$ exists and $\phi \to \int_{-\infty}^{\infty} u\phi\, dx$ is a distribution in $\mathcal{S}'$;

denote by $\hat{u}$ its Fourier transform in the sense of the theory of distributions.

PROPOSITION 4. *If* $u \in \mathfrak{L}^{(2)}$ *then* $\hat{u} \in L^2_{\mathrm{loc}}(R^1)$ *and the mapping* $\mathfrak{L}^{(2)} \ni u \to \hat{u} \in L^2_{\mathrm{loc}}$ *is continuous.*

PROOF. Let $\phi \in C_0^\infty[(-a, a)]$: denote $I_n = (n\pi/a, ((n+1)/a)\pi)$, $n = 0, \pm 1, \pm 2, \cdots$, and write

$$\hat{u}(\phi) = u(\hat{\phi}) = \sum_{n=-\infty}^{\infty} \int_{I_n} u\hat{\phi}dx = \sum_{n=-\infty}^{\infty} a_n. \tag{7}$$

Integrating by parts we get

$$\begin{aligned} a_n &= \int_{I_n} \hat{\phi}(x) d\left(\int_{n\pi/a}^{x} u(t)dt\right) \\ &= \hat{\phi}\left(\frac{n+1}{a}\pi\right)\int_{I_n} u(x)dx - \int_{I_n}\left(\int_{n\pi/a}^{x} u(t)dt\right)\frac{d\hat{\phi}(x)}{dx}dx, \end{aligned}$$

and by Schwartz's inequality

$$|a_n| \leqq \left|\hat{\phi}\left(\frac{n+1}{a}\right)\right|\left|\int_{I_n} udx\right| + \left(\frac{\pi}{a}\right)^{1/2}\int_{I_n}|u|\,dx\left(\int_{I_n}\left|\frac{d\hat{\phi}}{dx}\right|^2 dx\right)^{1/2}.$$

Using again the Schwartz inequality we get

$$\begin{aligned} \sum |a_n| \leqq &\left(\sum\left|\hat{\phi}\left(\frac{n+1}{a}\pi\right)\right|^2\right)^{1/2}\|u\|_{\pi/a} \\ &+ \left(\frac{\pi}{a}\right)^{1/2}\left(\int_{-\infty}^{\infty}\left|\frac{d\hat{\phi}}{dx}\right|^2 dx\right)^{1/2}\|u\|_{\pi/a}. \end{aligned} \tag{8}$$

By Parseval's identity the first term on the right-hand side of the last inequality is equal to $(a/\pi)^{1/2}\|\phi\|_{L^2(-a,a)}\|u\|_{\pi/a}$, the second is estimated, using Bernstein's inequality for entire functions of exponential type, by the expression $(a\pi)^{1/2}\|\phi\|_{L^2(-a,a)}\|u\|_{\pi/a}$. Combining this with (7) and (8) we get

$$|\hat{u}(\phi)| \leqq \left[\left(\frac{a}{\pi}\right)^{1/2} + (a\pi)^{1/2}\right]\|u\|_{\pi/a}\|\phi\|_{L^2(-a,\,a)} \tag{9}$$

which proves the proposition.

It follows from the last proposition that the integral transformation $\mathfrak{F}$ is $\mathfrak{L}^{(2)}$-semiregular and since $\mathfrak{L}^{(2)}$ is obviously an FL subspace of $\mathfrak{M}$ we get by virtue of Theorem 1 the inclusion $\mathfrak{L}^{(2)} \subset \tilde{\mathfrak{D}}_{\mathfrak{F}}$. Together with Proposition 2 this establishes the following

THEOREM 2. *The extended domain* $\tilde{\mathfrak{D}}_{\mathfrak{F}}$ *of the Fourier transform is identical with the space* $\mathfrak{L}^{(2)}$ *and* $\tilde{\mathfrak{F}}u = \hat{u}$ *for every* $u \in \mathfrak{L}^{(2)}$.

3. **Concluding remarks.** Define for $1 \leqq p < \infty$ and $u \in L^1_{\mathrm{loc}}(R^1)$

$$\|u\|_{a,p} = \left(\sum_{n=-\infty}^{\infty} \left(\int_{na}^{(n+1)a} |u|\, dx \right)^p \right)^{1/p}.$$

By the same argument as in Proposition 3 the norms $\|\ \|_{a,p}$ can be proved to be equivalent for all $a > 0$. An easy modification of the proof of Proposition 4 gives, with $\mathfrak{L}^{(p)}$ denoting the space of all functions u with $\|u\|_{\cdot,p}$ finite,

PROPOSITION 5. *If* $1 < p \leqq 2$ *and* $1/p + 1/p' = 1$, *then* $u \in \mathfrak{L}^{(p)}$ *implies* $u \in L^{p'}_{\mathrm{loc}}$ *and the mapping* $\mathfrak{L}^{(p)} \ni u \to \hat{u} \in L^{p'}_{\mathrm{loc}}$ *is continuous.*

The results of this paper can be extended to n-dimensional Fourier transform.

REFERENCES

1. N. Aronszajn and P. Szeptycki, *On general integral transformations*, Math. Ann. **163** (1966), 127–154.

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