THE DIRICHLET PROBLEM FOR NONUNIFORMLY ELLIPTIC EQUATIONS[1]

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Introduction. Let Ω be a bounded domain in E^n. The operator

$$Qu = a^{ij}(x, u, u_x)u_{x_ix_j} + a(x, u, u_x)$$

acting on functions $u(x) \in C^2(\Omega)$ is *elliptic* in Ω if the minimum eigenvalue $\lambda(x, u, p)$ of the matrix $[a^{ij}(x, u, p)]$ is positive in $\Omega \times E^{n+1}$. Here

$$u_x = (u_{x_1}, \cdots u_{x_n}), \qquad p = (p_1, \cdots p_n)$$

and repeated indices indicate summation from 1 to n. The functions $a^{ij}(x, u, p)$, $a(x, u, p)$ are defined in $\Omega \times E^{n+1}$. If furthermore for any $M > 0$, the ratio of the maximum to minimum eigenvalues of $[a^{ij}(x, u, p)]$ is bounded in $\Omega \times (-M, M) \times E^n$, Qu is called *uniformly elliptic.* A solution of the *Dirichlet problem* $Qu = 0$, $u = \phi(x)$ on $\partial\Omega$ is a $C^0(\bar{\Omega}) \cap C^2(\Omega)$ function $u(x)$ satisfying $Qu = 0$ in Ω and agreeing with $\phi(x)$ on $\partial\Omega$.

When Qu is elliptic, but not necessarily uniformly elliptic, it is referred to as *nonuniformly elliptic.* In this case it is well known from two dimensional considerations, that in addition to smoothness of the boundary data $\partial\Omega$, $\phi(x)$ and growth restrictions on the coefficients of Qu, geometric conditions on $\partial\Omega$ may play a role in the solvability of the Dirichlet problem. A striking example of this in higher dimensions is the recent work of Jenkins and Serrin [**4**] on the minimal surface equation, mentioned below.

The Dirichlet problem for general classes of nonuniformly elliptic equations has been considered by Gilbarg [**1**], Stampacchia [**7**], Hartman and Stampacchia [**2**], Hartman [**3**], and Motteler [**6**]. We announce below some theorems which extend the results of these authors. The detailed proofs will appear elsewhere.

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Equations of the form $a^{ij}(u_x)u_{x_ix_j} = 0$. Prior to stating our theorem we formulate a generalization of the well-known bounded slope condition, or B.S.C., used in [**2**], [**3**], and [**7**]. Let Γ be the $n-1$ dimen-

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sional manifold of $E^{n+1}(x, u)$ given by $\Gamma = \{x \in \partial\Omega,\ u = \phi(x)\}$, $\phi(x)$ defined on $\partial\Omega$. Hereafter we shall refer to Γ as the *boundary manifold* $(\partial\Omega, \phi)$ or simply as $(\partial\Omega, \phi)$.

DEFINITION 1. The boundary manifold $\Gamma = (\partial\Omega, \phi)$ satisfies a *generalized bounded slope condition* (G.B.S.C.) with respect to the operator $Qu = a^{ij}(u_x)u_{x_ix_j}$ if for all $P \in \Gamma$ there exists a neighborhood N_p of P and two functions $w^{\pm}(x) = w_p^{\pm}(x) \in C^2(\Omega \cap N_p) \cap C^{0,1}(\bar{\Omega} \cap N_p)$ satisfying

(i) $\pm Q(w^{\pm}) \leqq 0$ in $\Omega \cap N_p$,

(ii) $w^-(x) \leqq \phi(x) \leqq w^+(x)$, $x \in \partial\Omega \cap \overline{N}_p$;
$w^-(x) \leqq \min_{\partial\Omega} \phi(x)$, $\max_{\partial\Omega} \phi(x) \leqq w^+(x)$, $x \in \Omega \cap \partial N_p$,

(iii) the Lipschitz constants of the $w_p^{\pm}(x)$ are uniformly bounded, independently of P, by a constant R.

$C^{0,1}(S)$ denotes the space of functions uniformly Lipschitz continuous in S. R is called a constant of the G.B.S.C. In the special case $w_p^{\pm}(x) = \pi_p^{\pm}(x)$, $N_p \supset \Omega$ where $\pi_p^{\pm}(x)$ are planes passing through P, we have the usual B.S.C.

THEOREM 1. *Let Qu be elliptic, $a^{ij}(p) \in C^1(E^n)$ and $\partial\Omega$ satisfy an exterior sphere property. Then there exists a $C^2(\Omega) \cap C^{0,1}(\bar{\Omega})$ solution of the Dirichlet problem $Qu = 0$, $u = \phi(x)$ on $\partial\Omega$ if and only if the boundary manifold $(\partial\Omega, \phi)$ satisfies a G.B.S.C. with respect to Qu.*

Note that $\partial\Omega$ satisfies an *exterior sphere property* if for all $P \in \partial\Omega$, there exists a sphere S_p such that $\overline{S}_p \cap \bar{\Omega} = P$. It is probable that this assumption on $\partial\Omega$ may be removed from the hypotheses of Theorem 1.

We point out some special cases of Theorem 1. If the functions $a^{ij}(p)\lambda^{-1}(p)$ are bounded, i.e. Qu is uniformly elliptic, then a G.B.S.C. is satisfied if ϕ is the trace of a function with bounded second-order derivatives in Ω and $\partial\Omega$ satisfies a uniform exterior sphere property. If

$$Qu = (1 + |\nabla u|^2)\Delta u - u_{x_i}u_{x_j}u_{x_ix_j},$$

i.e. $Qu = 0$ is the minimal surface equation, then Jenkins and Serrin have proved that for arbitrary $\partial\Omega \in C^2$, $\phi(x) \in C^2(\partial\Omega)$, $(\partial\Omega, \phi)$ satisfies a G.B.S.C. if and only if the mean curvature of $\partial\Omega$ is of one sign [2]. This leads one to conjecture whether an analogous condition exists for general $a^{ij}(p)$. Such a condition would have to be void when Qu was uniformly elliptic.

Divergence structure equations. Assume that Qu has the form

$$Qu = \operatorname{div} \mathbf{a}(x, u, u_x) + a(x, u, u_x)$$

where $\mathbf{a}(x, u, u_x)$, $a(x, u, u_x)$ are respectively vector and scalar functions in $\Omega\times E^{n+1}$, div $= \sum\partial/\partial x_i$. When inhomogeneous terms are present in Qu, growth conditions on these terms are a factor in the solvability of the Dirichlet problem.

DEFINITION 2. Q satisfies the condition $P(\tau, \sigma)$ if for any $M>0$ the inequalities

$$\lambda(x, u, p) \geqq \lambda(|p|) > 0,$$

$$(1 + |p|)\,|\mathbf{a}_u(x, u, p)| + |\mathbf{a}_x(x, u, p)| + |a(x, u, p)| \leqq g(|p|)$$

hold in $\Omega\times(-M, M)\times E^n$ for some functions λ and g which are positive in $(0, \infty)$ and the functions $\lambda^*(t)=(1+t)^\tau\lambda(t)$, $g^*(t) =(1+t)^\tau g(t)$ are respectively nonincreasing and nondecreasing and satisfy

$$g^*(t) \leqq (1 + t)^\sigma\lambda^*(t).$$

If λ and g are independent of M, we shall say that Q satisfies $P(\tau, \sigma)$ uniformly in u.

In theorems on the Dirichlet problem for uniformly elliptic equations, Qu is assumed to satisfy $P(\tau, \sigma)$ for some $\tau\in E$, $\sigma\leqq 2$, [5], [8]. For nonuniformly elliptic equations we have

THEOREM 2. *Let Qu satisfy $P(\tau, \sigma)$ uniformly in u for some $\tau\in E$, $\sigma<1$. Let the coefficients of Qu be locally Hölder continuous in $\Omega\times E^{n+1}$, $\partial\Omega\in C^2$ be convex and suppose $(\partial\Omega, \phi)$ satisfies a B.S.C. Then the problem $Qu=\sigma$, $u=\phi$ on $\partial\Omega$ is solvable.*

For the particular case $\tau=0$, Theorem 3 has been proved by Hartman and Stampacchia [2], [3], Motteler [6]. In this case, we have in fact, the following extension of the theorems in [2], [3] and [6].

THEOREM 3. *Let Qu satisfy $P(0, \sigma)$ uniformly in u, and let the coefficients of Qu be locally Hölder continuous in $\Omega\times E^{n+1}$. Let $\partial\Omega$ be convex. Then if $\sigma\leqq 1$ the problem $Qu=0$, $u=0$ on $\partial\Omega$ is solvable. If $\sigma>1$, this problem is not necessarily solvable.*

The counterexample which demonstrates the last statement appears in [6]. Theorems 2 and 3 are also true under less severe restrictions on the behavior of the coefficients with respect to u.

We note in conclusion that Theorems 1, 2, and 3 possess parabolic analogues, i.e. analogues for equations of the form

$$Qu = a^{ij}(x, t, u, u_x)u_{x_ix_j} + a(x, t, u, u_x) - u_t = 0.$$

In the parabolic version of Theorem 3, there is no need for constant boundary values.

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