The similarity properties of Volterra operators on $L_p[0, 1]$ having reasonably smooth kernels seem to depend entirely on the behavior of the kernel as regards zeros and singularities on the diagonal $x = y$.

If $T_\sigma$ is a Volterra operator on $L_p[0, 1]$, then a study of its similarity properties seems to reduce to the following procedure involving the complex kernel $G(x, y)$.

1. Classify $G(x, x)$ according to its zeros and singularities on the interval $0 \leq x \leq 1$.

2. Show that $T_\sigma$ is similar to a unique $T_\nu$ for $T_\nu$ a canonical kernel of the class of which $G(x, y)$ belongs.

See [1], [2], and [4] for $G(x, y)$ of order $\alpha > 0$ i.e.

$$G(x, y) = (x - y)^{\alpha - 1}H(x, y)/\Gamma(\alpha)$$

with $H(x, x) > 0$ and $H(x, y)$ having certain smoothness properties.

The canonical form in this case is $KJ^\alpha$ for

$$K = \left[ \int_0^1 [H(t, t)]^{1/\alpha}dt \right]^{\alpha}$$

and

$$J^\alpha f = \int_0^x ((x - y)^{\alpha - 1}/\Gamma(\alpha))f(y)dy.$$ 

See [5] and [6] for $G(x, y)$ of rank 1, i.e.

$$G(x, x) < 0 \text{ if } 0 \leq x < x_0,$$

$$G(x, x) > 0 \text{ if } x_0 < x \leq 1,$$

$$G(x_0, x_0) = 0.$$

The canonical form in this case is $kQ_{a,\nu}$ for unique real $k, a$, and $\nu$ satisfying $0 \leq a, \nu \leq 1, 0 < k$.
\[ \int_0^1 G(t, t) dt = k \int_0^1 (x - a) dx = k/2(1 - 2a), \]

\[ \int_{x_0}^1 G(t, t) dt = k \int_{x_0}^1 (x - a) dx = k/2(1 - a)^2, \]

\[ G(x_0, x_0) / (G(x_0, x_0) + G(x_0, x_0)) = \nu, \]

and

\[ Q_{a, \nu} f = \int_0^\nu (x - a)'(t - a)^{1-\nu} f(t) dt. \]

These canonical forms are unique in a sense made precise in [4], [5], [7] and [8].

The important and delicate part of the previous work involved solving an equation of the form

\[ Q_{a, \nu} T_{\Gamma(B)} - T_{\Gamma(B)} Q_{a, \nu} = T_B \]

for the kernel \( \Gamma(B) \).

This is equivalent to an integral equation

\[ \int_0^\nu [(x - a)'(t - a)^{1-\nu} \Gamma(t, y) - \Gamma(x, t) (t - a)'(y - a)] dt = B(x, y). \]

Dupras, in his doctoral thesis [1], solved the commutator equation

\[ J^* \Gamma^{(a)}(B) - \Gamma^{(a)}(B) J^* = B \quad \text{for all } a > 0 \]

using a certain contour integral. He obtained the result

\[ \Gamma^{(a)}(B) = \int_0^{x-y} b(\sigma, y) d\sigma - (1/\alpha) \int_0^y b(x - y, t) dt + R_a(x, y) \]

for

\[ B(x, y) = \int_0^{x-y} ((x - y - \sigma)^a/\Gamma(\alpha + 1)) b(\sigma, y) d\sigma, \]

\[ R_a(x, y) \text{ is a certain contour integral depending on } \alpha. \]

Consider the Volterra integral operator

\[ Q[\alpha; a_1, a_2, \ldots, a_n; p_1, \ldots, p_n] = Q[\alpha; a_1, a_2, \ldots, a_n; p_1, \ldots, p_n] \]

which has the kernel \( F(x, y) \) such that

\[ F(x, y) = ((x - y)^{\alpha-1}/\Gamma(\alpha)) G(x, y) \]
with

\[ G(x, x) = \prod_{i=1}^{n} (x - a_i)^{p_i}, \quad 0 \leq a_1 < a_2 \ldots < a_n \leq 1, \]

\[ G_A(x, x) = \left( \prod_{j=1}^{n} (x - a_j)^{v_j} \right) \left( \sum_{i=1}^{n} v_i / (x - a_i) \right). \]

We shall later restrict \( \alpha, \nu_j \) and \( p_j \) in such a way that \( Q[\alpha, \alpha, \nu, \rho] \) is always a Volterra integral operator.

Suppose for some complex \( k \), we could get

\[ k M_{(1/1)} S_B J^a S_B M_1 = Q[\alpha, \alpha, \nu, \rho] \]

for \( S_B = f(S(x)) \) and \( M_1 = l(x)f(x) \) both bounded linear invertible transformations on \( L_{\rho}[0, 1] \) with \( S^{-1}(x) = R(x) \) or

\[ k u^{-1} J^a u = Q[\alpha, \alpha, \nu, \rho]. \]

It would then follow that the solution to the commutator equation

\[ [Q_{\alpha, \alpha, \nu, \rho}, T_X] = T_A \]

is

\[ T_X = u^{-1} T_\alpha (u T_\alpha) u^{-1} u / k \]

or for brevity

\[ X = u^{-1} \Gamma(\alpha) (u A u^{-1}) u / k. \]

In reality, the transformations \( u \) and \( u^{-1} \) will, in general, not be bounded, or for that matter well defined. However, we shall use this formalism to obtain candidates for \( X[\alpha, \alpha, \nu, \rho] \). In addition, the formalism yields a precise definition of \( Q[\alpha, \alpha, \nu, \rho] \).

The author used this method in [5] and obtained the canonical form for operators of rank one, i.e. the \( Q[\alpha, \alpha, \nu, \rho] \). The candidate for \( X[\alpha, \alpha, \nu, \rho] \) was found and was used to obtain the real commutator solution. See [5] and [6] for an exhaustive description of the similarity properties of operators of rank one.

It is known that a Volterra operator \( T_H \) with a reasonably smooth kernel \( H(x, y) \) commutes with \( J^\alpha \) iff \( H(x, y) = f(x - y) \).

Thus we would think

\[ [T_N, Q_{\alpha, \alpha, \nu, \rho}] = 0 \]

if

\[ T_N(x, y) = u^{-1} T_f(x - y) u. \]
We may use the operators which commute with $Q[a, a, v, p]$ to help make $X[a, a, v, p]$ well defined, i.e.

$$[Q[a, a, v, p], T X[a, a, v, p] + u^{-1} T f (x - y) u] = [Q[a, a, v, p], T X[a, a, v, p]].$$

Now we shall obtain the formal expressions for $k$, $S$, $l$, and $S^{-1} = R$.

$$k(R(x) - R(y))^{a-1} R'(y)(l(y)/l(x)) = (x - y)^{a-1} G(x, y),$$

(15)

$$k[R'(x)]^a = G(x, x) = \prod_{i=1}^n (x - a_i)^{p_i},$$

$$R(x) = \left( \int_0^x \prod_{i=1}^n (t - a_i)^{p_i/\alpha dt} \right)/k^{1/\alpha}$$

with

(16) $$k = \left[ \int_0^1 \prod_{i=1}^n (t - a_i)^{p_i/\alpha dt} \right]^\alpha$$

so that $R(1) = 1$.

(It is possible that $k = 0$. We ignore this difficulty and proceed formally.)

If we equate $x$ derivatives at $y = x$, we obtain

(17) $$l(x) = \prod_{i=1}^n (x - a_i)^{(1/2 - 1/2\alpha)p_i - r_i},$$

$$G(x, y) = \left( \left( \int_0^x \prod_{i=1}^n (t - a_i)^{p_i/\alpha dt} \right)/(x - y) \right)^{a-1}$$

(18)

$$\cdot \prod_{i=1}^n (x - a_i)^{r_i+p_i(1/2\alpha-1/2)}(y - a_i)^{-r_i+p_i(1/2\alpha+1/2)}.$$

In order that $T_F$ be a Volterra operator, we require

(19) $$\alpha \geq 1, \quad p_i(1/2 - 1/2\alpha) \leq n \leq p_i(1/2\alpha + 1/2), \quad 0 < p_i.$$

(This is not the most general case, but is sufficiently general for our purposes here.)

The corresponding commuting operator should be $T_N$ with

$$N(x, y) = f(R(x) - R(y)) R'(y) l(y)/l(x),$$

$$N(x, y) = \prod_{i=1}^n (x - a_i)^{r_i+p_i(1/2\alpha-1/2)}(y - a_i)^{p_i(1/2\alpha+1/2)-r_i}$$

(20)

$$\cdot f \left( \int_0^x \prod_{i=1}^n (t - a_i)^{p_i/\alpha dt} \right).$$
It can be shown formally that

\begin{equation}
Q_{a,a,v,p} \ast Q_{b,a,v,p} = Q_{a+b,a,v,p}.
\end{equation}

ACKNOWLEDGMENT. The author wishes to thank Professor J. T. Schwartz for his guidance in this work, which is based upon a doctoral dissertation at New York University.

BIBLIOGRAPHY


Brookhaven National Laboratory