A GEOMETRIC PROOF OF RYLL-NARDZEWSKI'S FIXED POINT THEOREM

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In [**4**], Ryll-Nardzewski gave what he called an 'old-fashioned' proof of his famous fixed point theorem. The purpose of the present note is to give an even more old-fashioned proof of the fixed point theorem. In fact, our proof uses nothing more than a category argument and the classical Krein-Milman theorem. Our terminology and notation shall be those of Kelley, Namioka et al. [**2**]. The following geometric lemma is essential to our proof of Ryll-Nardzewski's fixed point theorem. In case the space E and the pseudo-norm p in the lemma are a Banach space and its norm respectively, the lemma is an easy consequence of Lindenstrauss' work [**3**].[1]

LEMMA. *Let* $(E, \mathfrak{I})$ *be a locally convex Hausdorff linear topological space, let* K *be a nonempty* $\mathfrak{I}$*-separable, weakly compact, convex subset of* E, *and let* p *be a continuous pseudo-norm on* E. *Then for each* $\epsilon > 0$, *there is a closed convex subset* C *of* K *such that* $C \neq K$ *and* $p\text{-diam}(K \sim C) \leqq \epsilon$, *where, for any subset* X *of* E, $p\text{-diam}(X) = \sup\{p(x-y): x, y \in X\}$.

PROOF. Let $S = \{x: p(x) \leqq \epsilon/4\}$; then S is a weakly closed convex body. Let D be the weak closure of the set of all extreme points of K. Since K is $\mathfrak{I}$-separable, a countable number of translates of S cover K and hence D. Since D is weakly compact, it is of the second category in itself with respect to the relative weak topology. Therefore there are a point k of K and a weakly open subset W of E such that $(S+k)\cap D \supset W \cap D \neq \emptyset$. Let K_1 be the closed convex hull of $D \sim W$, and let K_2 be the closed convex hull of $D \cap W$. Then, by the Krein-Milman theorem and the compactness of K_1 and K_2, K is the convex hull of $K_1 \cup K_2$. Furthermore $K_1 \neq K$. For, otherwise, by Theorem 15.2 of [2], $D \sim W$ would contain all the extreme points of K, contradicting the fact that $W \cap D \neq \emptyset$. Obviously $p\text{-diam}(K_2) \leqq \epsilon/2$. Now let r be a real number in $(0, 1]$ and let f_r be the map $K_1 \times K_2 \times [r, 1] \to K$ defined by $f_r(x_1, x_2, \lambda) = \lambda x_1 + (1-\lambda)x_2$. Then clearly the image C_r of f_r is weakly closed, and it is easy to check that

[1] After the draft of the present note was completed we learned that Professor J. L. Kelley knew independently that a lemma of this sort was needed for a proof of Ryll-Nardzewski's fixed point theorem. Thus he was able to give a short proof of the fixed point theorem for Banach spaces using Lindenstrauss' result.

C_r is convex. Moreover $C_r \neq K$. For, if $C_r = K$, then each extreme point z of K is of the form $z = \lambda x_1 + (1-\lambda)x_2$, $x_i \in K_i$, $\lambda \in [r, 1]$. This would imply that each extreme point of K is in K_1 or $K = K_1$, contradicting $K_1 \neq K$. Finally, if $y \in K \sim C_r$, then y is of the form $y = \lambda x_1 + (1-\lambda)x_2$, $x_i \in K_i$, $\lambda \in [0, r)$. It follows that $p(y - x_2) = \lambda p(x_1 - x_2) \leqq rd$, where $d = p\text{-diam}(K) < \infty$. Since p-diam $(K_2) \leqq \epsilon/2$, we have $p\text{-diam}(K \sim C_r) \leqq \epsilon/2 + 2rd$. Therefore if we let $C = C_r$ for $r = \epsilon/4d$, the proof of the lemma is complete.

Let Q be a subset of a locally convex space E and let $\mathcal{S}$ be a semigroup of transformations of Q into Q. The semigroup $\mathcal{S}$ is called *noncontracting* if 0 does not belong to the closure of $\{Tx - Ty: T \in \mathcal{S}\}$ whenever $x \neq y$ and $x, y \in Q$. Clearly $\mathcal{S}$ is noncontracting if and only if, for $x, y \in Q$ with $x \neq y$, there is a continuous pseudo-norm p (depending on x and y) on E such that inf $\{p(Tx - Ty): T \in \mathcal{S}\} > 0$.

THEOREM (RYLL-NARDZEWSKI). *Let Q be a nonempty, weakly compact, convex subset of a locally convex Hausdorff linear topological space E, and let $\mathcal{S}$ be a noncontracting semigroup of weakly continuous affine maps of Q into itself. Then there is a common fixed point of $\mathcal{S}$ in Q.*

(The following proof is not the most direct one. However it establishes an additional interesting fact concerning fixed points, also due to Ryll-Nardzewski [4]: When $\mathcal{S}$ is finitely generated, the problem of finding a common fixed point of $\mathcal{S}$ can be reduced to that of a single operator.)

PROOF. By a familiar compactness argument, it is sufficient to prove that each finite subset of $\mathcal{S}$ has a common fixed point in Q. Therefore we may assume that $\mathcal{S}$ is generated by $T_1, T_2, \cdots, T_r$. Let $T_0 = (T_1 + T_2 + \cdots + T_r)/r$. Then T_0 is a weakly continuous affine map of Q into itself; hence there is a fixed point x_0 of T_0 in Q (see, for example, Théorème 1, Appendice of [1]). We will show that $T_i x_0 = x_0$ for $i = 1, \cdots, r$. Assume that this is not the case. Then by throwing out those T_i's for which $T_i x_0 = x_0$, we may assume that $T_i x_0 \neq x_0$ for $i = 1, 2, \cdots, r$.[2] Since $\mathcal{S}$ is noncontracting there is a continuous pseudo-norm p on E and $\epsilon > 0$ such that

$$(*) \qquad p(TT_i x_0 - Tx_0) > \epsilon \quad \text{for all } T \text{ in } \mathcal{S} \quad \text{and} \quad i = 1, \cdots, r.$$

Let K be the closed convex hull of $\{Tx_0: T \in \mathcal{S}\}$. Then K is a weakly compact, convex, separable subset of E. Hence, by the lemma, there is a closed convex subset C of K such that $C \neq K$ and

[2] Indeed if $T_i x_0 \neq x_0$ for $i \leq m$ and $T_i x_0 = x_0$ for $i > m$, then substitute $T_0' = (T_1 + \cdots + T_m)/m$ and the subsemigroup $\mathcal{S}'$ of $\mathcal{S}$ generated by $T_1, \cdots, T_m$ for T_0 and $\mathcal{S}$ respectively. Note that $T_0' x_0 = x_0$.

p-diam$(K\sim C)\leqq\epsilon$. Since $C\neq K$, there is an element S in $\mathcal{S}$ such that $Sx_0\in K\sim C$. From $T_0x_0=x_0$, we see that

$$Sx_0=(ST_1x_0+ST_2x_0+\cdots+ST_rx_0)/r.$$

Hence $ST_ix_0\in K\sim C$ for at least one i, since otherwise $Sx_0\in C$. It follows that $p(ST_ix_0-Sx_0)\leqq p$-diam$(K\sim C)\leqq\epsilon$, contradicting inequality (*). The proof of the theorem is therefore complete.

REMARK. In the proof above T_0 could have been any convex combination $\sum_{i=1}^{r}\lambda_iT_i$ with $\lambda_i>0$.

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