AN ISOMORPHISM PRINCIPLE IN GENERAL TOPOLOGY

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Introduction. To practically every topological space $T$ of importance (including metrizable and locally compact Hausdorff spaces) one can let correspond (essentially by interchanging compact and closed sets) an "antispace" $T^*$ which conversely determines $T$. If, for example, $T$ is Hausdorff but not compact, $T^*$ will be $T_1$, compact and superconnected (every open set is connected). One sacrifices the Hausdorff property but gains e.g. compactness. Furthermore the topology of $T^*$ is weaker than that of $T$. This destroys the belief, generally held, that non-Hausdorff spaces are of minor or no importance. On the contrary, one could even say that they are "more elegant," since they perform the same job with a weaker topology.

Philosophically, the consequences seem to be of interest. If $R$ denotes "time" (the real line), $R^*$ has the same topology as $R$ on every bounded closed interval. However $R^*$ is compact. Time becomes unbounded but finite in the sense of compact. We have potential but no actual infinity.

A remark by J. M. Aarts (in our joint work on cocompactness) initiated this note.

Preliminaries. Let $X$ be a set and $\{G\}$ a family of subsets $G$ of $X$, closed under finite unions and arbitrary intersections. We do not assume the (usual) convention that $X$ and $\emptyset$ are necessarily members of $\{G\}$. A pair $T_\sim = (X, \{G\})$ is called a (topological) minusspace, where $\{G\}$ indicates the family of closed sets of $T_\sim$. One can, of course, extend every $T_\sim$ to a topological space $T$ by adding $X$ and $\emptyset$ as closed sets.

A subset $S$ of $T_\sim$ is called squarecompact relative to $T_\sim$, if for every family $\{C_\alpha\}$ of compact subsets $C_\alpha$ of $T_\sim$, for which $\{S \cap C_\alpha\}$ is centered (that is the intersection of finitely many $S \cap C_\alpha$ is nonempty), the intersection of all $S \cap C_\alpha$ is nonempty.

One can prove:

(i) The intersection of a compact and a squarecompact set is both compact and squarecompact.

(ii) The union of finitely many and the intersection of any number of squarecompact sets is squarecompact.

(iii) If in $T_\sim$ every compact set is closed, then every closed set is squarecompact.
A topological space is called a \textit{c-space} if the closed sets are exactly those sets for which the intersection with every compact closed set is compact. This notion is different from the well-known notion of a \textit{k-space} (compactly generated space). However, every \textit{k}-space is a \textit{c-space} and for those spaces in which compact sets are closed, both notions coincide.

The following notions are equivalent for a topological space \( T \):

(a) It is a \textit{c-space}.
(b) Compact sets are closed and squarecompact sets are closed.
(c) Firstly, the intersection of a compact and a squarecompact set is closed; secondly, a set \( G \) is closed in \( T \) iff \( G \cap C \) is closed for all compact sets \( C \) of \( T \).
(d) Firstly, the intersection of two compact sets is compact; secondly, \( G \) is closed, iff \( G \cap C \) is compact for all compact \( C \).

Which spaces are \textit{c-spaces}?

(iv) In every locally compact topological space (that is, every point has arbitrarily small compact neighborhoods) or in every space satisfying the first axiom of countability the following properties are equivalent:

(a) Every compact set is closed,
(b) The space is a \textit{c-space},
(c) The space is Hausdorff.

\textbf{Antispaces.} Let \( X \) be a set, \( T_\_ = (X, \{ G \}, \{ C \}), T^* = (X, \{ C \}, \{ G \}) \) two minusspaces over \( X \), where \( \{ G \} \) denotes the family of all closed sets \( G \) and \( \{ C \} \) the family of all compact sets \( C \) in \( T_\_ \), while \( \{ C \} \) are the closed sets of \( T^* \) and \( \{ G \} \) the compact sets of \( T^* \). So the identity map of \( X \) onto itself maps the closed (compact) sets of \( X \) onto the compact (closed) sets of \( T^* \). Such a pair is called an antipair and \( T_\_ \) and \( T^* \) are called \textit{antispaces} (of each other). A space \( T_\_ \) is an antispaces, if there exists a \( T^* \) as indicated. Observe that antispaces determine each other. If in \( T_\_ \) the compact sets coincide with the closed sets, \( T_\_ \) and \( T^* \) coincide as e.g. is the case for compact Hausdorff spaces.

\textbf{Example.} If \( X \) is any set, and \( T \) the discrete space over \( X \), then \( T^* \) is determined by the cofinite topology on \( X \).

\textbf{Theorem.} A minusspace is an antispaces, iff the closed sets coincide with the squarecompact sets. The topological antispaces \( T \) are exactly the \textit{c-spaces}.

Observe that the \textit{compact} anti(minus)spaces \( T^* \) pair off with the \textit{topological} antispaces \( T \). A minusspace is a compact antispaces, iff every closed set is compact and the squarecompact sets coincide with
the closed sets. Notice that according to (iv), most spaces of importance in mathematics are c-spaces.

A category of antispaces $T$ and onto continuous mappings $f$ corresponds to the category of spaces $T^*$ and onto mappings $f^*$ ($f^*$ defined by the requirement, that the inverse of any compact image set is compact). This sets up an isomorphism as mentioned in the title.

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SOME EXPLICIT POLYNOMIAL APPROXIMATIONS
IN THE COMPLEX DOMAIN

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1. Let $f(z)$ be continuous on a compact set, $B$, in the complex plane. Then if $B$ contains more than $n$ points $f(z)$ has a unique best uniform approximation out of the polynomials of degree $n$. That is, if $P_n$ is the set of polynomials of degree at most $n$, there exists $p_n^* \in P_n$ satisfying

$$(1) \quad \| f - p_n^* \| < \| f - p \|$$

for all $p \in P_n$, $p \neq p_n^*$, the norm being the uniform norm. It is an instance of a result due to Kolmogorov (see Meinardus [2; p. 15], for example) that $p_n^* \in P_n$ satisfies (1) if, and only if,

$$(2) \quad \min_{\epsilon \in B} \Re [f(z) - p_n^*(z)] \geq 0, \quad p \in P_n,$$

where

$$(3) \quad E = \{ z/ \mid f(z) - p_n^*(z) \mid = \| f - p_n^* \| \}.$$  

Let us put

$$(4) \quad \rho_n(f; B) = \| f - p_n^* \|.$$  

In [1] Al'per showed that

$$(5) \quad \rho_n(z^p/(a^p - a^p); K_R) = R^{p+k+p^*}/(\mid a \mid^{2p} - R^{2p}) \cdot a \mid^{2k}$$