A NOTE ON MINIMAL VARIETIES[1]

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1. Introduction. In [1] Almgren considered the situation of a closed minimal variety H, of dimension 2 immersed in S^3. He observed that the second fundamental form, a real valued bilinear form on the tangent space to H, is in fact the real part of a holomorphic quadratic differential with respect to the conformal structure on H induced by the metric inherited from its immersion in S^3. He used this fact to conclude that S^2 could not be immersed as a minimal variety in S^3 unless it was already totally geodesic.

It turns out that under the most general circumstances the second fundamental form of a p-dim minimal subvariety of an n-dim Riemannian manifold satisfies a natural second-order elliptic differential equation which generalizes the holomorphic condition mentioned above. In the case that the ambient manifold is S^n the equation may be used to show that a closed minimal subvariety of S^n, of arbitrary codimension, which does not twist too much is already totally geodesic. In a sense this theorem is analogous to Bernstein's theorem for complete minimal subvarieties in R^n.

2. A standard operator. Let M be a Riemannian manifold[2] of dimension n and $V(M)$ a d-dimensional vector bundle over M. Suppose the fibers of $V(M)$ carry a euclidean inner product and suppose there is given a connection in $V(M)$ *which preserves this inner product.* If W is a cross-section in $V(M)$ and $x \in T(M)_m$, the tangent space to M at m, we denote by $\nabla_x W$ the covariant derivative of W in the x direction. $\nabla_x W \in V(M)_m$.

Let $x, y \in T(M)_m$. We define $\nabla_{x,y} W \in V(M)$ as follows. Let Y be a vector field on M which extends y. We then set

$$\nabla_{x,y} W = \nabla_x \nabla_Y W - \nabla_{\nabla_x Y} W \tag{2.1}$$

where $\nabla_x Y$ is ordinary covariant differentiation of a vector field on M with respect to the Riemannian connection. It is easy to see that this definition is independent of the choice of Y.

Let $e_1, \cdots, e_n$ be an orthonormal basis of $T(M)_m$. If W is a cross-section in $V(M)$ we define $\nabla^2 W$ by

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[2] All manifolds will be assumed to be orientable.

(2.2) $$\nabla^2 W = \sum_{i=1}^{n} \nabla_{e_i, e_i} W.$$

This definition of ∇^2 is independent of the choice of frame $e_1, \cdots, e_n$. Thus, ∇^2 is a second-order differential operator mapping the space of cross-sections of $V(M)$ into itself.

PROPOSITION 2.1. *∇^2 is an elliptic operator. If M is compact we have*

(2.3) $$\int_M \langle \nabla^2 W, Z\rangle = \int_M \langle W, \nabla^2 Z\rangle,$$

(2.4) $$\int_M \langle \nabla^2 W, W\rangle \leqq 0,$$

(2.5) $$\int_M \langle \nabla^2 W, W\rangle = 0 \Leftrightarrow \nabla^2 W = 0$$

$\Leftrightarrow W$ is covariant constant.

3. **The second fundamental form.** Let M be an n-dimensional C^∞ Riemannian manifold, H a p-dimensional manifold, and $\Phi: H \rightarrow M$ an immersion. We consider the following vector bundles over H: $T(H)$ = the tangent bundle; $N(H)$ = the normal bundle; $S(H)$ = the bundle of symmetric linear transformations of $T(H)_h \rightarrow T(H)_h$; $A(H) = \mathrm{Hom}(N(H), S(H))$. Each of these vector bundles has a natural euclidean inner product on its fibers, and each has a natural connection which preserves this inner product.

The second fundamental form. $\mathfrak{A}$ is a cross-section in $A(H)$. That is, for $w \in N(H)_h$, $\mathfrak{A}(w)$: $T(H)_h \rightarrow T(H)_h$ is a symmetric linear transformation. H is immersed as a *minimal variety* if and only if for each $h \in H$ and each $w \in N(H)_h$, $\mathrm{tr}\ \mathfrak{A}(w) = 0$.

$\mathfrak{A}$ gives rise to two natural linear maps at each point

$$\tilde{\mathfrak{A}}: N(H)_h \rightarrow N(H)_h; \qquad \boldsymbol{\mathfrak{A}}: S(H)_h \rightarrow S(H)_h$$

defined as follows. Since $N(H)_h$ and $S(H)_h$ are euclidean we may define $\mathfrak{A}^*$ = transpose of $\mathfrak{A}$. $\mathfrak{A}^*$: $S(H)_h \rightarrow N(H)_h$. We then set

$$\tilde{\mathfrak{A}} = \mathfrak{A}^* \circ \mathfrak{A}.$$

Let $f_1, \cdots, f_d$ be an orthonormal basis for $N(H)_h$, where $d = n - p$. We then set

$$\boldsymbol{\mathfrak{A}} = \sum_{i=1}^{d} (\mathrm{ad}(\mathfrak{A}(f_i)))^2.$$

This definition is independent of the choice of frame $\{f_i\}$.

Using $\mathfrak{a}$ and $\tilde{\alpha}$ we define $\bar{\alpha}(\alpha)$, a new cross-section in $A(H)$ by

$$\bar{\alpha}(\alpha) = \alpha \circ \tilde{\alpha} + \mathfrak{a} \circ \alpha.$$

Let R denote the curvature tensor of M. We use the convention that for $x, y \in T(M)_m$ and orthonormal, the sectional curvature, $k(x, y)$ of the plane section spanned by x and y satisfies $k(x, y) = -\langle R_{x,y}x, y\rangle$. By letting R operate on α we will construct a new cross-section, $R(\alpha)$, in $A(H)$.

For $x, y \in T(M)_{\phi(h)}$, $R_{x,y}$: $T(M)_{\phi(h)} \to T(M)_{\phi(h)}$ is a skew symmetric linear transformation. It induces:

$$\overset{N}{R}_{x,y}: N(H)_h \to N(H)_h,$$

$$\overset{T}{R}_{x,y}: T(H)_h \to T(H)_h,$$

$$\langle \overset{N}{R}_{x,y}z, w\rangle = \langle R_{x,y}z, w\rangle \quad z, w \in N(H)_h,$$

$$\langle \overset{T}{R}_{x,y}z, w\rangle = \langle R_{x,y}d\phi(z), d\phi(w)\rangle \quad z, w \in T(H)_h.$$

Then $R^N_{x,y}$ and $R^T_{x,y}$ are skew symmetric.

Let $e_1, \cdots, e_p$ be a frame in $T(H)_h$. Let $w \in N(H)_h$ and $x, y \in T(H)_h$. We define the cross-section, $R(\alpha)$, in $A(H)$:

$$\langle R(\alpha)(w)(x), y\rangle = \sum_{i=1}^{p} \left\{ \begin{array}{l} 2\langle \alpha(\overset{N}{R}_{x,e_i}w)(e_i), y\rangle + 2\langle \alpha(\overset{N}{R}_{y,e_i}w)(e_i), x\rangle \\ +\langle \alpha(\overset{N}{R}_{e_iw}e_i)(x), y\rangle - 2\langle \alpha(w)(e_i), \overset{T}{R}_{e_ix}y\rangle \\ -\langle \alpha(w)(x), \overset{T}{R}_{e_iy}e_i\rangle - \langle \alpha(w)(y), \overset{T}{R}_{e_ix}e_i\rangle \end{array} \right\}.$$

In the above expression, which is independent of the choice of $\{e_i\}$, we have sometimes identified points in $T(H)_h$ with points in $T(M)_{\phi(h)}$. E.g., $R^N_{x,e_i} = R^N_{d\phi(x),d\phi(e_i)}$.

Finally, we construct a third cross-section in $A(H)$ which exists independently of α. For $x \in T(M)_{\phi(h)}$ let $\nabla_x(R)$ denote the standard covariant derivative of the curvature tensor. We now define $R' \in A(H)_h$:

$$\langle R'(w)(x), y\rangle = \sum_{i=1}^{p} \left\{ \begin{array}{l} \langle \nabla_{e_i}(R)_{e_i,x}y, w\rangle \\ +\langle \nabla_{e_i}(R)_{e_i,y}x, w\rangle \\ +\langle \nabla_{w}(R)_{e_i,x}e_i, y\rangle \end{array} \right\}.$$

LEMMA 3.1. *If* $d = n - p = 1$, $\bar{\alpha}(\alpha) = \|\alpha\|^2\alpha$. *If* $d \geqq 2$, $0 \leqq \langle \bar{\alpha}(\alpha), \alpha\rangle \leqq \|\alpha\|^4$.

LEMMA 3.2. *If* $M = S^n$ *then* $R(\alpha) = p\alpha$ *and* $R' = 0$.

4. Minimal varieties.

THEOREM 4.1. *Let H be a C^∞ manifold of dimension p, M a C^∞ Riemannian manifold of dimension n, and ϕ: $H \rightarrow M$ an immersion. Suppose the image of H in M is a minimal variety. Then the second fundamental form, $\mathfrak{A}$, when regarded as a cross-section in the vector bundle $A(H)$ satisfies the equation*:

$$\nabla^2 \mathfrak{A} = -\bar{\mathfrak{A}}(\mathfrak{A}) + R(\mathfrak{A}) + R'. \tag{4.1}$$

THEOREM 4.2. *Let H be a C^∞ p-dimensional manifold immersed in S^n as a minimal variety. Then the second fundamental form, $\mathfrak{A}$ satisfies the equation*

$$\nabla^2 \mathfrak{A} = -\bar{\mathfrak{A}}(\mathfrak{A}) + p\mathfrak{A}. \tag{4.2}$$

COROLLARY 4.1. *Let H be a closed p-dimensional manifold immersed in S^n as a minimal variety. Then if at each point of H $\|\mathfrak{A}\|^2 < p$, H is totally geodesic, i.e., the image of H in S^n is the intersection of S^n with a p-dimensional subspace of R^{n+1}.*

THEOREM 4.3. *Let H be an immersed minimal variety of codimension 1 in S^n. Then the second fundamental form, $\mathfrak{A}$, satisfies the equation*

$$\nabla^2 \mathfrak{A} = (n - 1 - \|\mathfrak{A}\|^2)\mathfrak{A}. \tag{4.3}$$

Under the hypothesis of codimension 1 Formula (4.3) may be rewritten in a form which makes it subject to more careful analysis. Let V denote the unit normal vector field to H, chosen to make the orientation come out right. The second fundamental form, $\mathfrak{A}$, may now be regarded as a real valued symmetric bilinear form B, defined by

$$B(x, y) = \langle \mathfrak{A}(V)(x), y \rangle.$$

THEOREM 4.4. *Let H be an immersed minimal variety of codimension 1 in S^n. Let $\overline{R}$ denote the curvature of H with respect to the metric inherited from the immersion. Let $e_1, \cdots, e_{n-1}$ be a frame in $T(H)_h$. Then B satisfies the equation*

$$\nabla^2 B(x, y) = -\sum_{i=1}^{n-1} B(\overline{R}_{e_i,x}e_i, y) + B(e_i, \overline{R}_{e_i,x}y). \tag{4.4}$$

Equation (4.4) is interesting because both sides are defined intrinsically in terms of the geometry on H inherited from the immersion. The operator on the right-hand side is almost identical to the curvature operator on *skew symmetric* bilinear forms which appear as the linear piece of the Laplace-Beltrami operator. Although it is probably far from the best theorem, we can easily prove:

THEOREM 4.5. *Let g denote the standard metric on S^p. There exists a neighborhood of g in the space of nonequivalent Riemannian structure such that S^p together with any metric g′ in this neighborhood cannot be isometrically immersed in S^n as a minimal variety.*

Finally, we will express Equation (4.4) as a *first-order* condition on B and we will make the connection with holomorphic quadratic differentials mentioned in §1.

THEOREM 4.6. *Let B be a field of symmetric bilinear forms on a compact Riemannian manifold, H. Suppose* tr $B \equiv 0$. *Then B satisfies* (4.4) *if and only if B satisfies*

$$\nabla_x(B)(y, z) = \nabla_y(B)(x, z), \qquad \forall x, y, z \in T(H)_h. \tag{4.5}$$

If dim $H=2$, B satisfies (4.5) and tr $B=0$ if and only if the form $Q(x)=B(x, x)-iB(x, j(x))$ is a holomorphic quadratic differential (J being the usual 90° rotation). How to relate the dimension of the space of such forms on manifolds of higher dimension to some differential or geometric invariants seems to be a good problem.

BIBLIOGRAPHY

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