NONTRIVIAL m-INJECTIVE BOOLEAN ALGEBRAS
DO NOT EXIST

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We adopt the notation of Sikorski [3] with the following additions. A Boolean algebra \(\mathcal{A}\) is trivial iff it has only one element. \(\mathcal{A}\) is m-injective iff \(\mathcal{A}\) is an m-algebra and whenever we are given m algebras \(\mathcal{B}\) and \(\mathcal{C}\) with m-homomorphisms \(f, g\) of \(\mathcal{B}\) into \(\mathcal{A}\) and \(\mathcal{B}\) into \(\mathcal{C}\) respectively, and with \(g\) one-one, there is an m-homomorphism \(k\) of \(\mathcal{C}\) into \(\mathcal{A}\) such that \(f = k \circ g\) (\(\circ\) denotes composition of functions). Obviously every trivial Boolean algebra is m-injective for any m. Halmos [1] raised the question concerning what \(\sigma\)-injective Boolean algebras look like, and Linton [2] derived interesting consequences from the assumption that nontrivial \(\sigma\)-injectives exist.

The theorem of the title follows easily from the following two lemmas, the first of which is well known, while the second has some independent interest.

**Lemma 1.** If \(\mathcal{A}\) satisfies the m-chain condition, \(\{A_t\}_{t \in T}\) is a set of elements of \(\mathcal{A}\), and \(\bigcup_{t \in T} A_t\) exists, then there is a subset \(S\) of \(T\) with \(|S| \leq m\) such that \(\bigcup_{s \in S} A_s\) exists and equals \(\bigcup_{t \in T} A_t\).

**Proof.** Let \(\mathcal{B}\) be a maximal set of pairwise disjoint elements of \(\mathcal{A}\) such that for every \(B \in \mathcal{B}\) there is a \(t \in T\) such that \(B \subseteq A_t\) (such a \(\mathcal{B}\) exists by Zorn's lemma). With every \(B \in \mathcal{B}\) one can associate an element \(t(B)\) of \(T\) such that \(B \subseteq A_{t(B)}\). By the m-chain condition, \(|T| \leq m\), and hence also \(\{t(B)\}_{B \in \mathcal{B}}\) is m-indexed. Now \(\bigcup_{B \in \mathcal{B}} B\) exists and equals \(\bigcup_{t \in T} A_t\). For, if this is not true then, by virtue of the fact that \(B \subseteq \bigcup_{t \in T} A_t\) for each \(B \in \mathcal{B}\), it follows that there is a \(C \neq \Lambda\) such that \(B \cap C = \Lambda\) for all \(B \in \mathcal{B}\), while \(C \subseteq \bigcup_{t \in T} A_t\). Then \(C \cap A_{t_0} \neq \Lambda\) for a certain \(t_0 \in T\), and \(\mathcal{B} \cup \{C \cap A_{t_0}\}\) is a set properly including \(\mathcal{B}\) with all the properties of \(\mathcal{B}\). This contradiction shows that \(\bigcup_{B \in \mathcal{B}} B\) exists and equals \(\bigcup_{t \in T} A_t\). Obviously, then, \(\bigcup_{B \in \mathcal{B}} A_{t(B)}\) also exists and equals \(\bigcup_{t \in T} A_t\), as desired.

**Lemma 2.** For every \(m\) there is a complete Boolean algebra \(\mathcal{B}\) such that every nontrivial \(\sigma\)-homomorphic image of \(\mathcal{B}\) has cardinality at least \(m\).

**Proof.** Let \(\mathcal{B}\) be a free Boolean algebra on \(m\) generators, and let \(\mathcal{A}\) be a completion of \(\mathcal{B}\). By [3, pp. 72, 156], \(\mathcal{A}\) satisfies the \(\sigma\)-chain condition. Let \(I\) be a proper \(\sigma\)-ideal of \(\mathcal{A}\). By Lemma 1, \(I\) is principal;
say \( I \) is generated by \( A \subseteq \mathfrak{A} \). Then \( \mathfrak{A}/I \) is isomorphic to \( \mathfrak{A}/(-A) \) (see [3, pp. 30-31]); moreover, \( I \) proper implies that \( -A \neq V \). But \( \mathfrak{A} \) is homogeneous (see [3, pp. 106, 152]), and hence \( \mathfrak{A} \) is isomorphic to \( \mathfrak{A}/I \). Thus \( \mathfrak{A}/I \) has at least \( m \) elements, as desired.

**Theorem.** Nontrivial \( m \)-injective Boolean algebras do not exist.

**Proof.** Suppose that \( \mathfrak{A} \) is a nontrivial \( m \)-injective Boolean algebra. Let \( \mathfrak{B} \) be the two-element subalgebra of \( \mathfrak{A} \), and let \( \mathfrak{C} \) be a complete Boolean algebra such that every nontrivial \( \sigma \)-homomorphic image of \( \mathfrak{C} \) has power greater than \( \mathfrak{A} \). Let \( f \) and \( g \) be the natural isomorphisms of \( \mathfrak{B} \) and \( \mathfrak{A} \) and \( \mathfrak{C} \) respectively. By the \( m \)-injectiveness of \( \mathfrak{A} \) we obtain an \( m \)-homomorphism from \( \mathfrak{C} \) onto a nontrivial subalgebra of \( \mathfrak{A} \), which is impossible.

Linton has remarked to the author that this theorem can be improved to show that the category of \( m \)-algebras does not have a cogenerator.

**References**