THE UNIQUENESS OF THE (COMPLETE)
NORM TOPOLOGY

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In this paper we show that every semisimple Banach algebra over
\( \mathbb{R} \) or \( \mathbb{C} \) has the uniqueness of norm property, that is we show that if
\( \mathfrak{A} \) is a Banach algebra with each of the norms \( \| \cdot \|, \| \cdot \|' \) then these
norms define the same topology. This result is deduced from a maxi­
mum property of the norm in a primitive Banach algebra (Theorem 1).

In the following \( F \) is a field which may be taken throughout as \( \mathbb{R} \),
the real field, or \( \mathbb{C} \), the complex field. If \( \mathfrak{X} \) is a normed space then
\( \mathfrak{B}(\mathfrak{X}) \) will denote the space of bounded linear operators on \( \mathfrak{X} \).

**Lemma 1.** Let \( F, G \) be closed subspaces of the Banach space \( E \) such
that \( F+G=E \). Then there exists \( L>0 \) such that if \( x \in E \) then there is an
\( f \in F \) with

(i) \( \| f \| \leq L \| x \| . \)

(ii) \( x-f \in G. \)

**Proof.** The map \( (f, g) \rightarrow f+g \) is a continuous map of \( F \oplus G \) onto \( E \)
and is open by the open mapping theorem [1, p. 34]. Thus there is
\( \delta >0 \) such that if \( y \in E \) with \( \| y \| < \delta \) then there are \( f', g' \in G \) with
\( \| f' \|, \| g' \| \leq 1 \) and \( f'+g'=y \). The result of the lemma then follows if
we take \( L=\delta^{-1}, y=x\| x \|^{-1}\delta \) and \( f=f'\| x \| . \)

**Theorem 1.** Let \( \mathfrak{A} \) be a Banach algebra over \( F \) and let \( \mathfrak{X} \) be a normed
space over \( F \). Suppose that \( \mathfrak{X} \) is a faithful strictly irreducible left \( \mathfrak{A}-\)
module and that the maps \( \xi \rightarrow a\xi \) from \( \mathfrak{X} \) into \( \mathfrak{X} \) are continuous for each
\( a \in \mathfrak{A} \). Then there exists a constant \( M \) such that

\[ \| a\xi \|' \leq M \| a \| \| \xi \|' \]

for all \( a \in \mathfrak{A}, \xi \in \mathfrak{X}, \) where \( \| \cdot \| \) is the norm in \( \mathfrak{A} \) and \( \| \cdot \|' \) the norm in \( \mathfrak{X} \).

The theorem asserts that the natural map \( \mathfrak{A} \rightarrow \mathfrak{B}(\mathfrak{X}) \) is continuous.
It is a much stronger version of [4, Theorem 2.2.7] but applicable
only to primitive algebras. It would be interesting to know how far
it can be generalized.

**Proof.** If \( \xi \in \mathfrak{X} \) and \( a \rightarrow a\xi (\mathfrak{A} \rightarrow \mathfrak{X}) \) is continuous then the map \( a \rightarrow ab \rightarrow ab\xi \), being a composition of continuous maps, is continuous. Since
\( \mathfrak{X} \) is strictly irreducible, if \( \xi \neq 0 \) we can, by a suitable choice of \( b \), make
\( b\xi \) any particular vector in \( \mathfrak{X} \) and so if \( a \rightarrow a\xi \) is continuous for one
nonzero \( \xi \) it is continuous for all \( \xi \) in \( \mathfrak{X} \). We shall deduce a contradic-
tion by assuming \( a \to a\xi \) continuous only for \( \xi = 0 \) and hence show that all these maps are continuous. We assume \( \mathfrak{X} \neq \{ 0 \} \) since this case is trivial.

The \( \mathfrak{A} \)-module \( \mathfrak{X} \) is of infinite dimension over \( F \) since otherwise, as \( \mathfrak{X} \) is faithful, \( \mathfrak{U} \) would be a finite dimensional algebra and any linear map \( \mathfrak{A} \to \mathfrak{X} \) would be continuous. Since \( \mathfrak{X} \) is a strictly irreducible \( \mathfrak{A} \)-module the norm on \( \mathfrak{A} \) determines a complete norm \( \| \cdot \| \) on \( \mathfrak{X} \) [4, Theorem 2.2.6] and so the centralizer \( \mathfrak{D} \) of \( \mathfrak{A} \) on \( \mathfrak{X} \) is isomorphic with \( R, C \) or the quaternions [4, Lemma 2.4.4] and in any case is of finite dimension over \( F \). Since \( \mathfrak{X} \) is of infinite dimension over \( F \) it is of infinite dimension over \( \mathfrak{D} \). We can thus choose a linearly independent (over \( \mathfrak{D} \)) sequence \( \xi_1, \xi_2, \ldots \) from \( \mathfrak{X} \) with \( \| \xi_i \|' = 1 \).

We now show that for each \( K, \epsilon > 0 \) and for each positive integer \( m \) there is \( x \in \mathfrak{A} \) such that

(i)' \( \| x \| < \epsilon \).

(ii)' \( x\xi_1 = x\xi_2 = \cdots = x\xi_{m-1} = 0 \).

(iii)' \( \| x\xi_m \|' > K \).

Put \( J_i = \{ a; \ a \in \mathfrak{A}, a\xi_i = 0 \} \), then [3, p. 6, Theorem 2] \( J_i \) is a maximal modular left ideal and \( I = (J_1 \cap J_2 \cdots \cap J_{m-1}) + J_m \) is a left ideal containing \( J_m \). Since \( \xi_1, \ldots, \xi_m \) are linearly independent over \( \mathfrak{D} \) we can find, by the density theorem [3, p. 28], \( y \in \mathfrak{U} \) such that \( y\xi_1 = y\xi_2 = \cdots = y\xi_{m-1} = 0 \) and \( y\xi_m = \xi_m \neq 0 \). We have \( y \in I \), \( y \in J_m \) so that \( I \) contains \( J_m \) properly and, by maximality of \( J_m \), \( I = \mathfrak{A} \). Take the number \( L \) given by applying Lemma 1 with \( E = \mathfrak{U} \), \( F = J_1 \cap J_2 \cdots \cap J_{m-1} \), \( G = J_m \). By the discontinuity of the map \( x \to x\xi_m \) we can find \( x_0 \in \mathfrak{U} \) satisfying (i)' with \( \epsilon \) replaced by \( \epsilon/L \) and (iii)'. Then, by Lemma 1, there exists \( x \in J_1 \cap J_2 \cdots \cap J_{m-1} \) (so that (ii)' holds for \( x \)), such that \( x_0 - x \in J_m \) (i.e. \( x_0\xi_m = x\xi_m \)) and \( \| x \| \leq L \| x_0 \| < \epsilon \).

Now choose, by induction, a sequence \( x_1, x_2, \ldots \) in \( \mathfrak{A} \) such that

(i)'\( \| x_n \| < 2^{-n} \).

(ii)'\( x_n\xi_1 = \cdots = x_n\xi_{m-1} = 0 \).

(iii)'\( \| x_n\xi_m \|' \geq n + \| x_1\xi_n + \cdots + x_{n-1}\xi_n \|' \).

Put \( z_i = \sum_{n>i} x_n \). Since \( x_n \in J_i \) for \( n > i \) and \( J_i \) is closed in \( \mathfrak{A} \) we see that \( z_i \in J_i \), that is \( z_i\xi_i = 0 \), and \( z_0 = x_1 + \cdots + x_i + z_i \) Thus

\[
\| z_0\xi_i \|' = \| x_1\xi_i + \cdots + x_i\xi_i + z_i\xi_i \|' \\
\geq \| x_i\xi_i \|' - \| x_1\xi_i + \cdots + x_{i-1}\xi_i \|' \\
\geq i,
\]

using (iii)'\( \| \xi_i \|' = 1 \) this contradicts the hypothesis that \( \xi \to z_0\xi \) is a bounded linear operator in \( \mathfrak{X} \).

We have shown that \( (a, \xi) \to a\xi \) is continuous \( (\mathfrak{A}, \| \|) \to (\mathfrak{X}, \| \|') \)
for each $\xi \in \mathcal{F}$. The result of the theorem now follows since we also have that $(a, \xi) \rightarrow a\xi$ is continuous for fixed $a$ (by hypothesis) and so by [2, p. 38, Proposition 2] $(a, \xi) \rightarrow a\xi$ is jointly continuous.

**Theorem 2.** Let $\mathfrak{A}$ be a semisimple algebra over $\mathbb{R}$ or $\mathbb{C}$. Let $\| \|$, $\| \|'$ be norms on $\mathfrak{A}$ such that $(\mathfrak{A}, \| \|)$ and $(\mathfrak{A}, \| \|')$ are Banach algebras. Then the norms $\| \|$, $\| \|'$ define the same topology on $\mathfrak{A}$.

**Proof.** By [4, Chapter 2, §5, in particular p. 74] it is enough to prove the result for primitive $\mathfrak{A}$. Thus we are in the position of Theorem 1 with $\mathcal{X} = \mathfrak{A}/J$ for some maximal modular left ideal $J$ in $\mathfrak{A}$. We denote the quotient norms on $\mathcal{X}$ obtained from $\| \|$ and $\| \|'$ on $\mathfrak{A}$ by the same symbols. Suppose $\| x_n \| \rightarrow 0$ and $\| x_n - y \|' \rightarrow 0$ $(x_n, y \in \mathfrak{A})$. Then for each $\xi \in \mathcal{X}$ we have $\| x_n \xi - y \xi \|' \rightarrow 0$. However using Theorem 1 we see that $\| x_n \| \rightarrow 0$ implies $\| x_n \xi \|' \rightarrow 0$ so that $y \xi = 0$ for each $\xi \in \mathcal{X}$ and, since the representation is faithful, $y = 0$. The closed graph theorem [1, p. 37] then shows that the identity map $(\mathfrak{A}, \| \|) \rightarrow (\mathfrak{A}, \| \|')$ is continuous and the result follows by arguing with $\| \|$ and $\| \|'$ interchanged.

**References**


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