NOTE ON ARTIN'S SOLUTION OF HILBERT'S
17TH PROBLEM

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A uniquely orderable field \( F \) and a polynomial \( f(X) \) over \( F \) are constructed in such a manner that \( f(X) \), though positive at every point of \( F \), is not a sum of squares of elements of the rational function field \( F(X) \).

Artin's solution of Hilbert's problem asserts [2] that if a rational function assumes no negative values then it is a sum of squares, provided the coefficient field has exactly one order and that order is Archimedean; in Hilbert's formulation the coefficients are rational numbers. For definitions and a more detailed proof of Artin's theorem see Jacobson [6, Chapter VI]. Our example shows that the Archimedean hypothesis in Artin's theorem is not superfluous, contrary to Corollary 2, p. 278 of [8].

Let \( \mathbb{Q} \) be the field of all rational numbers, let \( t \) be an indeterminate, let \( \mathbb{Q}(t) \) be ordered so that \( t \) is positive and infinitesimal and let \( K \) be a real closure of \( \mathbb{Q}(t) \). Let \( F \) be the field over \( \mathbb{Q}(t) \) consisting of all elements of \( K \) obtainable from \( \mathbb{Q}(t) \) by means of a finite sequence of rational operations and square root extractions, exactly as in ruler and compass considerations. Since every positive element of \( F \) has its square roots in \( F \), \( F \) has exactly one order. Set [1, p. 115]

\[
f(X) = (X^2 - t)^2 - t^3,
\]

where \( X \) is a variable. Then \( f(X) \) is not a sum of squares in \( F(X) \) (nor even in \( K(X) \)), since \( f(1) \) and \( f(t^{1/3}) \) have opposite signs. Now we shall show that \( f(X) \) is positive as a function on \( F \). It has long been known [4], [7] that the ring \( B \) of all finite elements of \( K \) (\( u \) is finite if \( \left| u \right| < n \) for some integer \( n \)) is a valuation ring in \( K \). The induced valuation \( v \) is a measure of order of magnitude, the significance of \( v(a) < v(b) \) being that \( a^{-1}b \) is infinitesimal. Denoting by \( G \) the value group of \( K \) written in additive notation, and observing that \( G \) is a torsionfree abelian group, we shall show that \( G \) may be identified with (the additive group of) \( \mathbb{Q} \), with \( v(t) = 1 \). The ramification relation \( ef \leq n \) [3, p. 122], together with the algebraic character of \( K \) over \( \mathbb{Q}(t) \), implies that the rank of \( G \) is one. Hence [5, §42] \( G \) can be embedded in \( \mathbb{Q} \) so that \( v(t) \) maps onto \( 1 \); moreover \( K \) contains \( n \)th roots of \( t \) for all \( n \); so the embedding is onto. In other words \( G \) can be identified, and now will
be, with $Q$. It is altogether easy to see that if $f(y)$ is negative then $v(y) = 1/3$. But if $z$ is any member of $F$ then $z$ belongs to a field $H_r$ at the top of a finite tower

$$Q(t) = H_0 \subset H_1 \subset \cdots \subset H_r$$

of subfields of $K$, where each step is quadratic. An application of the ramification relation with $n = 2$ shows that the value group of $H_i$ has index one or two in the value group of $H_{i+1}$. Consequently $v(z)$ has the form $m/2^k$ for some integers $m$ and $k$. Since $m/2^k$ cannot be $1/3$, $f(z)$ is positive and all is proved.

References