RESIDUALLY FINITE ONE-RELATOR GROUPS

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Introduction. It seems to be commonly believed that the presence of elements of finite order in a group with a single defining relation is a complicating rather than a simplifying factor. This note is in support of the opposite point of view, lending respectability to the

CONJECTURE A. *Every group with a single defining relation with nontrivial elements of finite order is residually finite.*

In order to put our results in their proper setting let us define $\langle l, m\rangle$ to be the group generated by a and b subject to the single defining relation $a^{-1}b^l ab^m = 1$:

$$\langle l, m\rangle = (a, b;\ a^{-1}b^l ab^m = 1).$$

Adding a third parameter we define

$$\langle l, m; t\rangle = (a, b;\ (a^{-1}b^l ab^m)^t = 1).$$

Let $\mathcal{L}$ be the class of those groups $\langle l, m\rangle$ satisfying $|l| \neq 1 \neq |m|$, $lm \neq 0$, and l and m relatively prime. Furthermore, let $\mathfrak{M}$ be the class of these groups $\langle l, m; t\rangle$ satisfying the conditions imposed above on l and m, and in addition the extra two conditions $t > 1$, and l, m and t relatively prime in pairs. The point of our initial remark is that $\mathfrak{M}$ looks more complicated than $\mathcal{L}$. Actually $\mathcal{L}$ is quite a nasty class of groups. Indeed the main result of [1] is that every group in $\mathcal{L}$ is isomorphic to one of its proper factor groups, i.e. nonhopfian. Since finitely generated residually finite groups are hopfian (A. I. Mal'cev [2]) no group in $\mathcal{L}$ is residually finite. Our contribution to Conjecture A is that the groups in $\mathfrak{M}$ *are residually finite.*

THEOREM 1. *Every group in the class $\mathfrak{M}$ is residually finite.*

In fact even more is true.

THEOREM 2. *If l, m, t are relatively prime in pairs $(l \neq 0 \neq m)$ and if t is a power of a prime p $(t \neq 1)$ then the group $\langle l, m; t\rangle$ is residually a finite p-group.*

Conjecture A seems difficult. A somewhat easier related conjecture is

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CONJECTURE A$\flat$. *Every finitely generated group with a single defining relation with nontrivial elements of finite order is hopfian.*

The theory of groups with a single defining relation has been developed sufficiently for us to be able to prove

THEOREM 3. *Let G be a group with a single defining relation and let T be the subgroup of G generated by the elements of finite order. If G/T is hopfian, so is G.*

The existence of the nonhopfian group $\langle 2, 3\rangle$ together with Theorem 1 show that the converse of Theorem 3 is false. This underlines to some extent the difficulties involved in the proof of Theorem 1.

Remarks on the proofs. The proof of Theorem 1 goes as follows. Suppose $G\in\mathfrak{M}$. Thus

$$G = (a, b; (a^{-1}b^l ab^m)^t = 1).$$

We observe that if N is the normal closure of b in G then G/N is infinite cyclic. Our procedure is to prove that N is residually finite. Since an extension of a residually finite group by another residually finite group need not be residually finite we have to establish that N is residually finite in such a way that we are able to deduce the residual finiteness of G. To establish the results we need about N we have to obtain sufficient information about certain one-relator subgroups from which N is constructed. This information is contained in the following lemmas.

LEMMA 1. *The groups*

$$(a, b; (a^l b^m)^t = 1) \qquad (t > 1)$$

contain a normal subgroup of finite index which is residually free.

LEMMA 2. *The groups*

$$(a, b; (a^l b^m)^t = 1) \qquad (t > 1)$$

are residually finite p-groups if t is a power of the prime p.

Both Lemma 1 and Lemma 2 make use of the Reidemeister-Schreier procedure for finding generators and defining relations for a subgroup of a group given by generators and defining relations (see [**3**, p. 86]) as well as the main results of [**4**] and [**5**] on the residual properties of certain generalized free products.

The proof of Theorem 2 involves a refinement of the proof of Theorem 1 and an old theorem of P. Hall, namely that an automorphism of a finite p-group P which induces an automorphism of p-power order on P modulo its frattini subgroup is itself of p-power order (see e.g. [**6**, p. 178]).

Finally the proof of Theorem 3 depends on the known structure of T [**7**] and the fact that in a one-relator group every pair of elements of maximal finite order are conjugate [**8**].

Extension of results. Theorem 1 can be extended to certain groups with a single defining relation on more than two generators. At the present time I am unable to relax the conditions on l, m and t to $t>1$. But it is certainly likely that $\langle l, m; t\rangle$ $(t>1)$ is residually finite. This can probably be proved by similar arguments to those used in the proof of Theorem 1. A proof of Conjecture A, however, at this time, seems out of reach.

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