

SOME GROUPS THAT ARE JUST ABOUT FREE

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Introduction. I have recently come across a rather extraordinary class of groups \mathfrak{G} while looking for a group of cohomological dimension 1 which is not free. The groups $G \in \mathfrak{G}$ are generated by three elements a, b, c satisfying the single defining relation $a = c^{-i}a^{-1}c^i a \cdot c^{-j}b^{-1}c^j b$:

$$(1) \quad G = \langle a, b, c; a = c^{-i}a^{-1}c^i a c^{-j}b^{-1}c^j b \rangle \quad (ij \neq 0).$$

The purpose of this announcement is to make known the surprisingly similar behavior of these groups in \mathfrak{G} and the free group F of rank two.

THEOREM. *Every group G in \mathfrak{G} satisfies the following conditions:*

(i) G (like F) is the third term of an exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Z \rightarrow 1$$

where N is free and Z is infinite cyclic;

(ii) the 2-generator subgroups of G (like those of F) are free;

(iii) G (like F) is residually nilpotent, i.e.

$$\bigcap_{i=1}^{\infty} \gamma_i G = 1,$$

where $\gamma_i G$ is the i th term of the lower central series of G ;

(iv) $G/\gamma_i G \cong F/\gamma_i F$ for $i = 1, 2, \dots$;

(v) $G/G'' \cong F/F''$ where X'' is the second derived group of the group X ;

(vi) G is not free.²

Before making a few remarks about the proof of the Theorem I would like to point out that, by a theorem in [1], G is of cohomological dimension at most 2. Whether every group in \mathfrak{G} is of cohomological dimension precisely 2, I do not as yet know!

Incidentally, groups satisfying (iii) and (iv) (termed parafree in [2]) are plentiful [2]. The main point of the theorem is that nonfree parafree groups G satisfying (v) can exist.

Remarks on the proof of the theorem. Let G be the group given by (1). We verify that G has the properties (i)–(vi).

¹ The author is a Sloan Fellow.

² I thank S. Meskin for helping to verify (vi).

(i) is straightforward and is proved by taking N to be the normal closure of a and b and applying the Reidemeister-Schreier procedure for finding generators and relations for a subgroup of a group given by generators and relations (see e.g. [3, p. 86]).

(ii) holds for all parafree groups [4].

(iii) is the most difficult property to verify. One proves that G has this property by making use of the exact sequence (i).

(iv) and (v) follow exactly from the observation that G is the freest group generated by a , b and c satisfying the relation $a^{-1}c^{-i}a^{-1}c^i a c^{-j} b^{-1} c^j b = 1$.

(vi) There is an algorithm introduced by J. H. C. Whitehead [5] whereby one can effectively determine whether a group with a single defining relation is free. (vi) follows on applying this algorithm to G .

REFERENCES

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