SUBDIRECT PRODUCTS OF SEMIGROUPS AND RECTANGULAR BANDS

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Let $\{S_\alpha : \alpha \in A\}$ be a family of semigroups. If p_α is the natural projection from $\Pi\{S_\alpha : \alpha \in A\}$ onto S_α, then a subsemigroup D of $\Pi\{S_\alpha : \alpha \in A\}$ is called a subdirect product of $\{S_\alpha : \alpha \in A\}$ if $p_\alpha(D) = S_\alpha$ for all $\alpha \in A$.

If L and R are sets then the semigroup $B = L \times R$ with $(\lambda_1, \rho_1) \cdot (\lambda_2, \rho_2) = (\lambda_1, \rho_2)$ is called a rectangular band. Our main result, Theorem 1, determines all subdirect products of a semigroup S and a rectangular band B. Elements of $S \times B$ will be denoted by $(s; \lambda, \rho)$ $(s \in S, \lambda \in L, \rho \in R)$.

Proofs of the following results will appear elsewhere. See [1] for all undefined concepts.

THEOREM 1. *Let S be a semigroup and $B = L \times R$ be a rectangular band. If $\mathcal{L}$ is the set of all left ideals of S and $\mathcal{R}$ is the set of all right ideals of S, then two mappings $\phi: L \to \mathcal{R}$ and $\psi: R \to \mathcal{L}$ satisfying*

$$S = \bigcup\{\phi(\lambda) : \lambda \in L\} = \bigcup\{\psi(\rho) : \rho \in R\}$$

determine a subdirect product $D \subseteq S \times B$ by

$$D = \bigcup\{D(\lambda, \rho) : (\lambda, \rho) \in B\},$$

where

$$D(\lambda, \rho) = \{(x; \lambda, \rho) : x \in \phi(\lambda) \cap \psi(\rho)\}.$$

Moreover, the correspondence $(\phi, \psi) \to D$ is one-to-one onto the set of all subdirect products of S and B.

One application of this theorem is

COROLLARY 1. *Let S be a semigroup and $B = L \times R$ be a rectangular band. The only subdirect product of S and B is the direct product of S and B if and only if one of the following is satisfied:*

(i) *S is right simple, and B is a left zero semigroup, $B \cong L$,*

(ii) *S is left simple, and B is a right zero semigroup, $B \cong R$,*

(iii) *S is a group, or*

(iv) *B is trivial, $|B| = 1$.*

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We now consider an isomorphism problem. Suppose D_i is a subdirect product of S_i and B_i ($i=1, 2$). If $D_1 \cong D_2$, what can we say about S_1, S_2, B_1 and B_2? When S_1 and S_2 are commutative we can conclude that $B_1 \cong B_2$ but not that $S_1 \cong S_2$. Restricting S_1 and S_2 further we have

THEOREM 2. *For $i=1, 2$, let D_i be a subdirect product of a rectangular band B_i and a semigroup S_i which is commutative and reductive ($ax=bx$ for all $x \in S_i$ implies $a=b$). If $D_1 \cong D_2$, then $B_1 \cong B_2$ and $S_1 \cong S_2$.*

We also determine results concerning subdirect products of s-indecomposable semigroups. A semigroup S is s-indecomposable if any semilattice homomorphic image Y of S is trivial, $|Y|=1$.

It can be proved (see [2] and [3]) that s-indecomposability is preserved by finite direct products, but examples can be given to show that this property is not preserved by either infinite direct products or finite subdirect products. As a special case we have

THEOREM 3. *All subdirect products of a semigroup S and rectangular band are s-indecomposable if and only if S is s-indecomposable.*

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