GLOBAL CONTINUOUS SOLUTIONS OF HYPERBOLIC SYSTEMS OF QUASI-LINEAR EQUATIONS[1]

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Recently there have appeared a number of results on global solutions of the Cauchy problem for hyperbolic systems of quasi-linear equations [2], [3], [4], [5]. These solutions are in general discontinuous. In certain cases, however, such as the interaction of two rarefaction waves in gas dynamics, it is known that the Cauchy problem has a global continuous solution [1, pp. 191–197]. In this announcement we outline a proof that a global continuous solution exists and is unique for a two-dimensional system provided the Riemann invariants associated with the initial data satisfy certain monotonicity and continuity conditions.

Let $\lambda^+(r, s)$, $\lambda^-(r, s)$ be C^1 real-valued functions on a domain $D \subset R_2$, with

$$\lambda^+(r, s) > \lambda^-(r, s), \qquad \partial\lambda^+(r, s)/\partial r > 0, \qquad \partial\lambda^-(r, s)/\partial s > 0 \tag{1}$$

for $(r, s) \in D$. Consider the two-dimensional system of quasi-linear equations in Riemann invariant form

$$r_t + \lambda^+(r, s) r_x = 0, \qquad s_t + \lambda^-(r, s) s_x = 0 \tag{2}$$

where $r(t, x)$ and $s(t, x)$ are real-valued functions of two scalar variables. We seek a solution of the Cauchy problem in the halfplane $\{(t, x) \in R_2 : t \geqq 0\}$ with initial conditions

$$r(0, x) = r^0(x), \qquad s(0, x) = s^0(x), \qquad -\infty < x < +\infty. \tag{3}$$

Let $G_T = \{(t, x) \in R_2 : 0 \leqq t < T\}$ for $0 < T \leqq +\infty$. A pair of Lipschitz continuous functions $(r(t, x), s(t, x))$, $(t, x) \in G_T$, is called a Lipschitz continuous solution of the Cauchy problem (2), (3) if $r(t, x)$ is constant on the integral curves

$$x'(t) = \lambda^+(r(t, x), s(t, x)), \tag{4}$$

$s(t, x)$ is constant on the integral curves

$$x'(t) = \lambda^-(r(t, x), s(t, x)), \tag{5}$$

[1] These results are part of the author's Ph.D. thesis written under Joel A. Smoller at the University of Michigan.

and $r(0, x)$, $s(0, x)$ satisfy the initial conditions (3). The pair $(r(t, x), s(t, x))$ is called a global Lipschitz continuous solution of (2), (3) if the functions are defined and Lipschitz continuous on G_∞.

THEOREM 1. *If $r^0(x)$, $s^0(x)$, $-\infty < x < +\infty$, are bounded, Lipschitz continuous, and nondecreasing, satisfying*

$$[r^0(-\infty), r^0(+\infty)] \times [s^0(-\infty), s^0(+\infty)] \subset D, \tag{6}$$

then the Cauchy problem (2), (3) *with initial functions $r^0(x)$, $s^0(x)$, $-\infty < x < +\infty$, has a global Lipschitz continuous solution which takes its values in the rectangle* (6).

OUTLINE OF PROOF FOR THEOREM 1. For each finite subset A of R_1 we construct an approximate solution $(r(t, x; A), s(t, x; A))$, $(t, x) \in G_\infty$, with the property that $r(t, x; A)$ is constant on curves of the form (4) and $s(t, x; A)$ is constant on a finite number of curves of the form (5). Using condition (1) and the assumed properties of the initial functions, we show that $r(t, x; A)$ is Lipschitz continuous in G_∞ with Lipschitz constant independent of t, x, and A. If $\{B_n : n = 1, 2, \cdots\}$ is an increasing sequence of finite sets whose union is dense in R_1, then by the Ascoli theorem the sequence of functions $r(t, x; B_n)$ contains a subsequence converging to a Lipschitz continuous function $r(t, x)$. Having this function, we construct $s(t, x)$, $(t, x) \in G_\infty$, so that the pair $(r(t, x), s(t, x))$, $(t, x) \in G_\infty$, is a global Lipschitz continuous solution of (2), (3).

We have also obtained the following result regarding the dependence of Lipschitz continuous solutions on initial data.

THEOREM 2. *Let $r_i^0(x)$, $s_i^0(x)$, $-\infty < x < +\infty$, $i = 1, 2$, be bounded real-valued functions with*

$$a_i \leqq r_i^0(x) \leqq b_i, \qquad c_i \leqq s_i^0(x) \leqq d_i, \qquad -\infty < x < +\infty,$$

and suppose that $[a_i, b_i] \times [c_i, d_i] \subset D$, $i = 1, 2$. Let

$$m = \sup_{-\infty < x < +\infty} \left(\left| r_1^0(x) - r_2^0(x) \right| + \left| s_1^0(x) - s_2^0(x) \right| \right).$$

If $(r_i(t, x), s_i(t, x))$, $(t, x) \in G_T$, is a Lipschitz continuous solution of the Cauchy problem for the equations (2) *with initial vector $(r_i^0(x), s_i^0(x))$, $-\infty < x < +\infty$, $i = 1, 2$, then there is a constant $L(T)$ such that*

$$\sup_{(t,x) \in G_T} \left(\left| r_1(t, x) - r_2(t, x) \right| + \left| s_1(t, x) - s_2(t, x) \right| \right) \leqq mL(T).$$

It follows easily from this that Lipschitz continuous solutions are unique.

Theorems 1 and 2, together with results of Lax [3], can be applied to the conservation law

$$u_t + (p(v))_x = 0, \qquad v_t - u_x = 0 \tag{7}$$

to yield the following corollaries.

COROLLARY 1. *Let* $p(v) \in C^2$ *on the halfline* $(v>0)$ *with* $p'(v)<0$, $p''(v)>0$ *and* $\int_1^\infty [-p'(v)]^{1/2}dv = \infty$. *If the functions* $u^0(x)$, $v^0(x)$, $-\infty < x < +\infty$, *are bounded and Lipschitz continuous, with* $v^0(x)$ *positive and bounded away from* 0, *and satisfy*

$$u^0(x_2) - u^0(x_1) \geqq \left| \int_{v^0(x_1)}^{v^0(x_2)} [-p'(v)]^{1/2} dv \right| \qquad \text{for } x_2 > x_1, \tag{8}$$

then the Cauchy problem for the equations (7) *with initial vector* $(u^0(x), v^0(x))$ *has a unique Lipschitz continuous weak solution.*

COROLLARY 2. *Let* $p(v)$ *be as in Corollary* 1, *and let* $u^0(x)$, $v^0(x)$, $-\infty < x < +\infty$, *be bounded and piecewise constant real-valued functions, with* $v^0(x)$ *positive and bounded away from* 0, *which satisfy* (8). *If the set* A *of discontinuities of the vector function* $(u^0(x), v^0(x))$ *has the property*

$$\inf\{ |a - b| : a, b \in A, a \neq b\} > 0,$$

then the Cauchy problem for the system (7) *with initial vector* $(u^0(x), v^0(x))$ *has a solution which is Lipschitz continuous in each of the sets* $\{(t, x) \in G_\infty : t > t_0\}$, for $t_0 > 0$.

Corollary 2 provides a solution for the interaction of simple waves centered on the line $(t=0)$.

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