GLOBAL CONTINUOUS SOLUTIONS OF HYPERBOLIC SYSTEMS OF QUASI-LINEAR EQUATIONS

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Recently there have appeared a number of results on global solutions of the Cauchy problem for hyperbolic systems of quasi-linear equations [2], [3], [4], [5]. These solutions are in general discontinuous. In certain cases, however, such as the interaction of two rarefaction waves in gas dynamics, it is known that the Cauchy problem has a global continuous solution [1, pp. 191–197]. In this announcement we outline a proof that a global continuous solution exists and is unique for a two-dimensional system provided the Riemann invariants associated with the initial data satisfy certain monotonicity and continuity conditions.

Let \( \lambda^+(r, s), \lambda^-(r, s) \) be \( C^1 \) real-valued functions on a domain \( D \subset \mathbb{R}^2 \), with

\[
(1) \quad \lambda^+(r, s) > \lambda^-(r, s), \quad \frac{\partial \lambda^+(r, s)}{\partial r} > 0, \quad \frac{\partial \lambda^-(r, s)}{\partial s} > 0
\]

for \( (r, s) \in D \). Consider the two-dimensional system of quasi-linear equations in Riemann invariant form

\[
(2) \quad r_t + \lambda^+(r, s) r_x = 0, \quad s_t + \lambda^-(r, s) s_x = 0
\]

where \( r(t, x) \) and \( s(t, x) \) are real-valued functions of two scalar variables. We seek a solution of the Cauchy problem in the halfplane \( \{(t, x) \in \mathbb{R}^2 : t \geq 0\} \) with initial conditions

\[
(3) \quad r(0, x) = r^0(x), \quad s(0, x) = s^0(x), \quad -\infty < x < +\infty.
\]

Let \( G_T = \{(t, x) \in \mathbb{R}^2 : 0 \leq t < T\} \) for \( 0 < T \leq +\infty \). A pair of Lipschitz continuous functions \((r(t, x), s(t, x))\), \((t, x) \in G_T\), is called a Lipschitz continuous solution of the Cauchy problem (2), (3) if \( r(t, x) \) is constant on the integral curves

\[
(4) \quad x'(t) = \lambda^+(r(t, x), s(t, x)),
\]

\( s(t, x) \) is constant on the integral curves

\[
(5) \quad x'(t) = \lambda^-(r(t, x), s(t, x)),
\]

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and \( r(0, x), s(0, x) \) satisfy the initial conditions (3). The pair \((r(t, x), s(t, x))\) is called a global Lipschitz continuous solution of (2), (3) if the functions are defined and Lipschitz continuous on \( G_\infty \).

**Theorem 1.** If \( r^0(x), s^0(x), -\infty < x < +\infty, \) are bounded, Lipschitz continuous, and nondecreasing, satisfying

\[
[r^0(-\infty), r^0(+\infty)] \times [s^0(-\infty), s^0(+\infty)] \subset D,
\]
then the Cauchy problem (2), (3) with initial functions \( r^0(x), s^0(x), -\infty < x < +\infty, \) has a global Lipschitz continuous solution which takes its values in the rectangle (6).

**Outline of Proof for Theorem 1.** For each finite subset \( A \) of \( R_1 \) we construct an approximate solution \((r(t, x; A), s(t, x; A))\), \((t, x) \in G_\infty\), with the property that \( r(t, x; A) \) is constant on curves of the form (4) and \( s(t, x; A) \) is constant on a finite number of curves of the form (5). Using condition (1) and the assumed properties of the initial functions, we show that \( r(t, x; A) \) is Lipschitz continuous in \( G_\infty \) with Lipschitz constant independent of \( t, x \), and \( A \). If \( \{B_n: n = 1, 2, \ldots\} \) is an increasing sequence of finite sets whose union is dense in \( R_1 \), then by the Ascoli theorem the sequence of functions \( r(t, x; B_n) \) contains a subsequence converging to a Lipschitz continuous function \( r(t, x) \). Having this function, we construct \( s(t, x) \), \((t, x) \in G_\infty, \) so that the pair \((r(t, x), s(t, x)), (t, x) \in G_\infty, \) is a global Lipschitz continuous solution of (2), (3).

We have also obtained the following result regarding the dependence of Lipschitz continuous solutions on initial data.

**Theorem 2.** Let \( r_i^0(x), s_i^0(x), -\infty < x < +\infty, i = 1, 2, \) be bounded real-valued functions with

\[
a_i \leq r_i^0(x) \leq b_i, \quad c_i \leq s_i^0(x) \leq d_i, \quad -\infty < x < +\infty,
\]
and suppose that \([a_i, b_i] \times [c_i, d_i] \subset D, i = 1, 2.\) Let

\[
m = \sup_{-\infty < x < +\infty} (| r_1^0(x) - r_2^0(x) | + | s_1^0(x) - s_2^0(x) |).
\]

If \((r_i(t, x), s_i(t, x)), (t, x) \in G_T, \) is a Lipschitz continuous solution of the Cauchy problem for the equations (2) with initial vector \((r_i^0(x), s_i^0(x)), -\infty < x < +\infty, i = 1, 2, \) then there is a constant \( L(T) \) such that

\[
\sup_{(t, x) \in G_T} (| r_1(t, x) - r_2(t, x) | + | s_1(t, x) - s_2(t, x) |) \leq mL(T).
\]
It follows easily from this that Lipschitz continuous solutions are unique.

Theorems 1 and 2, together with results of Lax [3], can be applied to the conservation law
\[ u_t + (p(v))_x = 0, \quad v_t - u_x = 0 \]
to yield the following corollaries.

**Corollary 1.** Let \( p(v) \in C^2 \) on the halfline \( (v > 0) \) with \( p'(v) < 0, \ p''(v) > 0 \) and \( \int_1^\infty \left[ -p'(v) \right]^{1/2} dv = \infty. \) If the functions \( u^0(x), \ v^0(x), \ -\infty < x < +\infty, \) are bounded and Lipschitz continuous, with \( v^0(x) \) positive and bounded away from 0, and satisfy
\[ I \left[ -p'(v) \right]^{1/2} dv \] for \( x_2 > x_1, \)
then the Cauchy problem for the equations (7) with initial vector \( (u^0(x), \ v^0(x)) \) has a unique Lipschitz continuous weak solution.

**Corollary 2.** Let \( p(v) \) be as in Corollary 1, and let \( u^0(x), \ v^0(x), \ -\infty < x < +\infty, \) be bounded and piecewise constant real-valued functions, with \( v^0(x) \) positive and bounded away from 0, which satisfy (8). If the set \( A \) of discontinuities of the vector function \( (u^0(x), \ v^0(x)) \) has the property
\[ \inf \{ | a - b | : a, b \in A, a \neq b \} > 0, \]
then the Cauchy problem for the system (7) with initial vector \( (u^0(x), \ v^0(x)) \) has a solution which is Lipschitz continuous in each of the sets \( \{(t, x) \in G_0 : t > t_0\} \), for \( t_0 > 0. \)

Corollary 2 provides a solution for the interaction of simple waves centered on the line \( (t = 0). \)

**References**


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