A SUBALGEBRA OF $\mathrm{Ext}_A^{**}(Z_2, Z_2)$

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Let A be the mod 2 Steenrod algebra and

$$H^{**}(A) = \mathrm{Ext}_A^{**}(Z_2, Z_2)$$

its cohomology. $H^{s,t}(A)$ has been computed up to certain values of $t-s$ by Adams [2], Ivanovski (International Congress of Mathematicians, Moscow (1966)), Liulevicius (unpublished), May [4], [5], Tangora [8]. It is of interest to know any "systematic" phenomena in $H^{**}(A)$.

It is the object of this note to sketch a simple proof of the following result.

THEOREM. *$H^{**}(A)$ contains a subalgebra generated by the elements*

$$d_0 \in H^{4,18}(A), \quad e_0 \in H^{4,21}(A), \quad g \in H^{4,24}(A)$$

subject to the single relation $e_0^2 = d_0 g$. That is the elements $e_0^i d_0^j g^k$ with $i=0, 1, j \geqq 0, k \geqq 0$ are linearly independent.

The notation d_0, e_0, g is taken from May [4].

Since obtaining this proof, the author became aware of the work of Mahowald and Tangora [7]. The theorem given above is contained in [7] (except for certain low-dimensional cases, which are clearly known to Mahowald and Tangora). However the author hopes that the present line of proof can be pushed further than is done here.

The line of proof depends on choosing a suitable subalgebra B of A. We take B to be the exterior subalgebra generated by $Sq^{0,1}$ and $Sq^{0,2}$. Then $H^{**}(B)$ is a polynomial algebra on two generators, namely $x = \{[\xi_2]\}$ and $y = \{[\xi_2^2]\}$. (Here we have used the notation of the cobar construction [1].) The inclusion $i: B \to A$ induces a map

$$i^{**}: H^{**}(A) \to H^{**}(B).$$

The proof depends on showing that

$$i^{**}d_0 = x^2y^2, \quad i^{**}e_0 = xy^3, \quad i^{**}g = y^4.$$

This evidently shows that the elements $e_0^i d_0^j g^k$ with $i=0, 1, j \geqq 0, k \geqq 0$ are linearly independent. To obtain the relation $e_0^2 = d_0 g$, we observe that by the above argument, e_0^2 and $d_0 g$ are both nonzero elements of $H^{8,42}(A)$; but by [4, Appendix A], $H^{8,42}(A) = Z_2$. Thus $e_0^2 = d_0 g$. This proves the theorem.

It remains to sketch how to compute the effect of i^{**} on d_0, $\tilde{e}_0$ and g. The inclusion $i\colon B\rightarrow A$ induces a known map $i^*\colon A^*\rightarrow B^*$ of the dual algebras and a map

$$F(i^*): F(A^*) \rightarrow F(B^*)$$

of the cobar construction. To deal with e_0, for example, it is now sufficient to exhibit an explicitly cocycle $\tilde{e}_0$ of bidegree 4, 21 in $F(A^*)$ such that

$$\{F(i^*)\tilde{e}_0\} = xy^3.$$

(Since $H^{4,21}(A)=Z_2$ [4], it follows that $\{\tilde{e}_0\}=e_0$.) We construct e_0 by constructing a representative cocycle for the quadruple Massey product $\langle h_3^2, h_0^2, h_1, h_0\rangle$. Similarly, we construct explicit cocycles d_0 of bidegree 4, 18 and g of bidegree 4, 24. The constructions involve known relations between the classes h_i (see [1]); they also use Steenrod $\cup_i$ operations in $F(A^*)$.

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