UNIFORMLY BOUNDED REPRESENTATIONS OF SL(2, C)

BY RONALD L. LIPSMAN[1]

Communicated by N. Levinson, May 5, 1967

1. **Introduction.** In [1] Kunze and Stein construct a family of continuous representations of SL(2, R) with the following properties: the representations act on a *fixed* Hilbert space; they are indexed by a complex parameter, and depend analytically on that parameter; included among them are the principal and complementary series representations; and they are uniformly bounded. By applying suitable convexity and Phragmén-Lindelöf type arguments to these bounds and the Plancherel formula for the group, they derive some important applications to harmonic analysis on SL(2, R).

Later, in [2], Kunze and Stein construct a family of representations of SL(n, C) having the same properties as described above (except that they depend analytically on $n-1$ complex variables). However, the uniform bounds obtained are not sufficient to prove any results concerning harmonic analysis on SL(n, C). The author has modified their construction, in the case of $G=\mathrm{SL}(2, C)$, so that it more closely resembles the method used in [1]. As a result, one obtains much sharper estimates on the uniform bounds. One of the consequences is the remarkable fact: Convolution by an $L_p(G)$ function, $1\leqq p<2$, is a bounded operator on $L_2(G)$.

2. **Uniformly bounded representations of** SL(2, C). Consider the multipliers

$$\phi(g, z, n, s) = (\beta z+\delta)^{-n}\,|\beta z+\delta|^{n-2-2s},$$
$$g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in G = \mathrm{SL}(2, C), \quad z\in C, \quad n \in Z$$

and $s=\sigma+it$ a complex number.
Define the multiplier representations $g\to T(g, n, s)$, given for f on the complex plane, by

$$T(g, n, s): f(z) \to \phi(g, z, n, s) f((\alpha z+\gamma)/(\beta z+\delta)).$$

Then the nontrivial irreducible unitary representations of G are:

(a) Principal series: $g\to T(g, n, it)$, $n\in Z$, $t\in R$, $f\in L_2(C)$;

[1] N.S.F. graduate fellow.

(b) Complementary series: $g \to T(g, 0, \sigma)$, $0 < \sigma < 1$,

where the Hilbert space is defined by the inner product

$$(f_1, f_2)_\sigma = a_\sigma \int_C \int_C |z_1 - z_2|^{-2+2\sigma} f_1(z_1) [f_2(z_2)]^- dz_1 dz_2, \qquad a_\sigma > 0.$$

THEOREM 1. *There exists a family of representations* $g \to R(g, n, s)$ *of* G *on* $L_2(C)$ *such that*:

(1) $g \to R(g, n, s)$ *is a continuous representation for each* s, $-1 < \operatorname{Re}(s) < 1$, *each* $n \in Z$;

(2) $g \to R(g, n, it)$ *is unitarily equivalent to* $g \to T(g, n, it)$, $n \in Z$, $t \in R$;

(3) $g \to R(g, 0, \sigma)$ *is unitarily equivalent to* $g \to T(g, 0, \sigma)$, $0 < \sigma < 1$;

(4) *If* $\psi_1, \psi_2 \in L_2(C)$, *then* $s \to (R(g, n, s)\psi_1, \psi_2)$ *is analytic in* $-1 < \operatorname{Re}(s) < 1$, g *and* n *fixed*;

(5) *For any* $\epsilon > 0$,

$$\sup_{g \in G} \|R(g, n, s)\|_\infty \leqq A_{\sigma,\epsilon}(1 + |n| + |t|)^{|\sigma|(1+\epsilon)}.$$

Moreover, for any fixed $\epsilon > 0$, the numbers $A_{\sigma,\epsilon}$ are uniformly bounded on the intervals $\sigma \in [\alpha, \beta]$, $-1 < \alpha < \beta < 1$.

We remark only that the proof is very similar to that of theorem 5 of [**1**].

3. **Harmonic analysis on** SL(2, C). If A is a bounded operator on a Hilbert space and $1 \leqq p < \infty$, let $\|A\|_p = [\text{trace } (A^*A)^{p/2}]^{1/p}$. For a discussion of the Banach spaces $\mathfrak{B}_p = \{A : \|A\|_p < \infty\}$, see [**1**, §2]. Let $\|A\|_\infty$ denote the usual norm as a bounded operator. Then the Plancherel formula for G is:

$$\|f\|_2^2 = \left(\frac{1}{2\pi}\right)^4 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \|R(f, n, it)\|_2^2 (n^2 + 4t^2) dt \qquad f \in (L_1 \cap L_2)(G),$$

where $R(f, n, s) = \int_G R(g, n, s) f(g) dg$. (See [**3**, p. 225]; there, Naimark's representation $\mathfrak{S}_{m,\rho}(g)$ corresponds to our $T(g, -m, -i\rho/2)$.) Also, part (5) of Theorem 1 implies:

$$\sup_{t \in R} (1 + |t|)^{-|\sigma|(1+\epsilon)} \|R(f, n, \sigma + it)\|_\infty \leqq A_{\sigma,\epsilon} |n|^{|\sigma|(1+\epsilon)} \|f\|_1.$$

Then by Theorem 4 of [**1**], which interpolates on the complex numbers $\sigma + it$ as well as the L_p parameters, $1 \leqq p \leqq 2$, we can prove

THEOREM 2. *Let* $1 < p < 2$, $1/p + 1/q = 1$. *Let* $s = \sigma + it$ *with* $1/q - 1/p$

$<\sigma<1/p-1/q$ *and* $\sigma\neq 0$. *Then for every* $\delta>0$, *there exists* $\epsilon>0$ *and constants* $A_{p,\sigma,\epsilon}$ *such that*

$$\left(\int_{-\infty}^{\infty}\|R(f,n,s)\|_q^q(1+|t|)^{1-q|\sigma|-\delta}dt\right)^{1/q}$$

$$\leqq A_{p\sigma,\epsilon}|n|^{|\sigma|\tau(1+\epsilon)/(1-\tau)+1/2(\tau-1)}\|f\|_p$$

for all simple functions f. τ *is given by* $1/p=1-\tau/2$, $0<\tau<1$; *and if* $n=0$, *the power of* $|n|$ *is understood to be* 1.

Theorem 2, which resembles the Hausdorff-Young theorem for abelian groups, is the key to the conclusions we wish to draw about harmonic analysis on G. By restricting to narrower strips, we can make the exponent of $|n|$ nonpositive. Then by general Phragmén-Lindelöf arguments (analogous to [1, §8]), we get: For each p, $1\leqq p<2$,

$$\sup_{n,t}\|R(f,n,it)\|_\infty \leqq A_p\|f\|_p,\quad f\in L_p.$$

As an immediate consequence of this fact and the Plancherel formula for G, we obtain

THEOREM 3. *Let* $f\in L_2(G)$, $h\in L_p(G)$, $1\leqq p<2$. *If* $k(g)=(f*h)(g) = \int_G f(gg_0^{-1})h(g_0)dg_0$, *then* $k\in L_2(G)$ *and* $\|k\|_2\leqq A_p\|f\|_2\|h\|_p$. *That is, the operation of convolution by an* L_p *function*, $1\leqq p<2$, *is a bounded operator on* $L_2(G)$.

4. **Fourier transform on** SL(2, C). For $f\in L_1(G)$, define the Fourier transform F to be the measurable operator-valued function $F(n,s) = R(f,n,s) = \int R(g,n,s)f(g)dg$. It is an interesting consequence of Theorem 2 that the Fourier transform extends to all of $L_p(G)$, $1\leqq p<2$, as an analytic operator-valued function $F(n,s)$ in the strip $1-2/p<\mathrm{Re}(s)<2/p-1$, ($n$ fixed). Furthermore, using Phragmén-Lindelöf arguments again, we can prove a Riemann-Lebesgue lemma for G which is stronger than its classical analog.

THEOREM 4. *Let* $f\in L_p$, $1\leqq p<2$, $1/p+1/q=1$, *and let* F *be the Fourier transform of* f. *Then*

(1) $\|F(n,it)\|_\infty$ *vanishes at infinity in the sense of the plane topology*, $Z\times R\subseteq R^2$;

(2) *For any strip* $-d\leqq\mathrm{Re}(s)\leqq d$, $0\leqq d<\min(1/q, 1/p-1/q)$, *we have* $\|F(n,s)\|_\infty\to 0$ *uniformly as* $|t|\to\infty$. *Here* n *is fixed and for* $p=1$, *the only strip we consider is the line* $\mathrm{Re}(s)=0$.

Finally make the

DEFINITION. A unitary representation U of a locally compact group H is extendible to $L_p(H)$, $p>1$, if $\|U(f)\|_\infty \leqq A\|f\|_p$ for all

$$f\in(L_1\cap L_p)\ (H).$$

THEOREM 5. *Let $g\to U_g$ be an irreducible unitary representation of G. Assume U is not the identity representation. Then*

(1) *U is unitarily equivalent to an element of the principal series if and only if U is extendible to every $L_p(G)$, $1\leqq p<2$.*

(2) *U is unitarily equivalent to the element of the complementary series corresponding to the parameter σ, $0<\sigma<1$ if and only if U is extendible to every $L_p(G)$, $1\leqq p<2/(1+\sigma)$, but it is not extendible to $L_{2/(1+\sigma)}(G)$.*

BIBLIOGRAPHY

1. R. A. Kunze and E. M. Stein, *Uniformly bounded representations and harmonic analysis of the* 2×2 *real unimodular group*, Amer. J. Math. **82** (1960), 1–62.

2. ———, *Uniformly bounded representations.* II. *Analytic continuation of the principal series of representations of the* $n\times n$ *complex unimodular group*, Amer. J. Math. **83** (1961), 723–786.

3. M. A. Naimark, *Linear representations of the Lorentz group*, Macmillan New York, 1964.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY