SEMIHEREDITARY RINGS

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1. **Introduction.** A ring is (right) semihereditary if every finitely-generated right ideal is projective. Chase [2] gave the first example of a ring which was right but not left semihereditary. In [7] the author constructed an example of a ring which was even right hereditary (all right ideals are projective) yet not left semihereditary.

In the other direction, P. M. Cohn, [3] and [4], has found certain classes of rings for which right semihereditary implies left semihereditary. In particular, total matrix rings over principal right ideal domains are both right and left semihereditary. In this note, among other things, we shall show that if a ring is right Noetherian and right hereditary then it is also left semihereditary.

2. **Notation and definitions.** Ring means ring with identity element, and all modules are unitary. R_n will denote the ring of all $n \times n$ matrices over the ring R. If S is a subset of the ring R, then $r(S)$ ($l(S)$) will denote the right (left) annihilator of S.

3. **Principal results.** The following sublemma is well known, and we omit the proof.

SUBLEMMA. *The following statements are equivalent:*

(1) *The ring R has no infinite set of orthogonal idempotents.*

(2) *The right (left) ideals of the form eR (Re) where e is an idempotent satisfy the ascending and descending chain conditions.*

We recall that the right ideal aR is projective if and only if $r(a) = eR$ where e is an idempotent.

THEOREM 1. *Let R be a ring in which every principal right ideal is projective and in which there is no infinite set of orthogonal idempotents. Then every right and every left annihilator is generated by an idempotent. In particular, every principal left ideal is projective.*

PROOF. Suppose $(0) \neq T = r(S)$. If $s \in S$, then $r(s) \supseteq T$. Thus, $T \subseteq hR$ where h is an idempotent. Now let L be an arbitrary (nonzero) left annihilator. $r(L) \subseteq gR$ where $g^2 = g$. But then $L = l(r(L)) \supseteq l(gR) = R(1-g)$. Hence, any left annihilator, L, contains a nontrivial idempotent.

The Sublemma allows us to pick an idempotent $e \in L$ such that $l(e)$ is minimal amongst the left annihilators of idempotents in L. We claim $l(e) \cap L = (0)$. Suppose not. Then $l(e) \cap L \neq (0)$ and is a left annihilator which contains an idempotent $f \neq 0$, say. Now $e^* = e + f - ef$ is an idempotent in L. Since $e^* e = e$, $e^* \neq 0$ and $l(e^*) \subseteq l(e)$. However, $fe = 0$ and $fe^* = f \neq 0$. Thus, $l(e^*) \not\subseteq l(e)$, which contradicts the minimality of $l(e)$. Hence, $l(e) \cap L = (0)$. Now if $x \in L$, then $x - xe \in L$ and $(x - xe)e = 0$. Therefore $x - xe = 0$ and $L = Re$. Finally, if K is a right annihilator, then $l(K) = Re$ where $e^2 = e$. But, $K = r(l(K)) = (1 - e)R$.

Corollary 1. *If R is as in the previous theorem, then R satisfies the ascending and descending chain conditions on left and right annihilators. Furthermore, if N is the maximal nil ideal of R, then N is nilpotent and contains all nil right and left ideals.*

Proof. The first statement follows from Theorem 1 and the Sublemma. The second statement follows from a result of Herstein and the author [5].

We state without comment

Corollary 2. *If R is a right perfect ring* [1] *in which principal right ideals are projective, then R is semiprimary and principal left ideals are projective. In addition, a right hereditary, right perfect ring is also left hereditary.*

Of course, no right Noetherian ring has an infinite set of orthogonal idempotents; hence, for such rings principal right ideals projective implies principal left ideals projective.

The next proposition provides the key to extending Theorem 1 to the case of semihereditary rings.

Proposition. *A ring R is right* (*left*) *semihereditary if and only if R_n, for all n, has principal right* (*left*) *ideals projective.*

Proof. It is well known that if R is right (left) semihereditary, then so is R_n.

In the other direction, we must show that any finitely-generated right ideal, say $I = a_1R + \cdots + a_nR$, is projective. In R_n let x be the matrix (c_{ij}) where $c_{1i} = a_i$ and all other entries are zero. Then xR_n is projective as a right R_n-module. But, xR_n considered as a right R-module (R embedded in R_n in the usual way) is isomorphic to $I \oplus \cdots \oplus I$ (n times). Thus, since R_n is R-free, $I \oplus \cdots \oplus I$ is R-projective and I is R-projective.

Combining the Proposition and Theorem 1, we immediately obtain

THEOREM 3. *Suppose R is a ring which is right semihereditary and such that R_n, for all n, does not possess an infinite set of orthogonal idempotents, then R is left semihereditary.*

Although only a special case, we single out

COROLLARY 3. *If R is right Noetherian and right hereditary, then R is left semihereditary.*

We remark that both Theorem 3 and Corollary 3 are "best possible." In [6] the author constructed an example of a right Noetherian, right hereditary ring which was not left hereditary. Furthermore, the examples of rings which are semihereditary on the right but not on the left have infinite sets of orthogonal idempotents.

We record the following corollaries without proof.

COROLLARY 4. *Let R be right Noetherian and right hereditary. If e is an idempotent in R, then e Re, is also right hereditary.*

COROLLARY 5. *If R is as in the previous corollary and every left ideal of R requires no more than $\aleph_n$, $n < \infty$, generators, the left global dimension of R is at most $n+2$.*

This last corollary generalizes a result of C. Jensen (to appear). We close with the observation that if R is a ring which has a homomorphism f, with kernel in the Jacobson radical, to a ring T with the property that T_n, for all n, has no infinite set of orthogonal idempotents then R has the same property.

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