WIENER-HOPF OPERATORS AND ABSOLUTELY CONTINUOUS SPECTRA[1]

BY C. R. PUTNAM

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If A is a selfadjoint operator on a Hilbert space $\mathfrak{H}$ with spectral resolution $A = \int \lambda dE_\lambda$, it is known that the set of elements x in $\mathfrak{H}$ for which $\|E_\lambda x\|^2$ is an absolutely continuous function of λ is a subspace, $\mathfrak{H}_a(A)$, reducing A; cf. Halmos [1, p. 104]. In case $\mathfrak{H}_a(A) = \mathfrak{H}$, A is said to be absolutely continuous. The following was proved in Putnam [4]; see also [5] and will be stated as a

LEMMA. *Let T be a bounded operator on a Hilbert space $\mathfrak{H}$ and let*

$$(1) \qquad T^*T - TT^* = C, \qquad C \geqq 0.$$

If $A = T + T^*$, then $\mathfrak{H}_a(A) \supset \mathfrak{M}_T$, where $\mathfrak{M}_T$ is the least subspace of $\mathfrak{H}$ reducing T (that is, invariant under T and T^*) and containing the range of C.

The above will be used to give a short proof of the absolute continuity of certain bounded selfadjoint Wiener-Hopf operators on $L^2(0, \infty)$. For an extensive account of Wiener-Hopf operators on the half-line see Krein [2].

Let $k(t)$, for $-\infty < t < \infty$, satisfy

$$(2) \qquad k \in L^1(-\infty, \infty) \cap L^2(-\infty, \infty) \quad \text{and} \quad k(-t) = \bar{k}(t).$$

Then the operator T on $\mathfrak{H} = L^2(0, \infty)$ defined by

$$(3) \qquad (Tf)(t) = \int_0^t k(s-t)f(s)ds, \qquad 0 \leqq t < \infty,$$

is bounded. (In fact, the hypothesis $k \in L^1(-\infty, \infty)$ alone implies the boundedness of T, even $\|T\| \leqq \int_{-\infty}^{\infty} |k(t)| dt$; cf. Krein [2, pp. 201–202].) The adjoint T^*, which is given by

$$(4) \qquad (T^*f)(t) = \int_t^\infty k(s-t)f(s)ds,$$

and the selfadjoint operator $A = T + T^*$, where

$$(5) \qquad (Af)(t) = \int_0^\infty k(s-t)f(s)ds,$$

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are of course also bounded on $\mathfrak{H}=L^2(0, \infty)$. There will be proved the following

THEOREM. *Let* $k(t)$ *satisfy* (2) *and suppose that*

$$K(\lambda) \equiv \int_{-\infty}^{\infty} k(t)e^{i\lambda t}dt \neq 0 \text{ a.e.}, \qquad -\infty < \lambda < \infty. \tag{6}$$

Then the bounded selfadjoint operator A *of* (5) *is absolutely continuous.*

PROOF. A calculation similar to that in Putnam [3, p. 517], shows that, for $f\in\mathfrak{H}$, $\|Tf\|^2-\|T^*f\|^2=\|Bf\|^2$, where T is defined in (3) and

$$(Bf)(t) = \int_0^{\infty} k(t+s)f(s)ds, \tag{7}$$

so that (1) holds with $C=B^*B$. It will be shown that the set $\mathfrak{M}_T$ of the Lemma is, in the present case, the entire space $\mathfrak{H}=L^2(0, \infty)$, and hence the absolute continuity of A will follow.

For $f\in L^2(0, \infty)$, define the Fourier transform $\hat{f}(\lambda)$ and the functions $F_+(\lambda)$ and $F_-(\lambda)$ by

$$\hat{f}(\lambda) = \int_0^{\infty} e^{-i\lambda t}f(t)dt \equiv F_-(\lambda) \tag{8}$$

and

$$F_+(\lambda) = \int_0^{\infty} e^{i\lambda t}f(t)dt. \tag{9}$$

The space of elements F_+ is a subspace of $L^2(-\infty, \infty)$ and will be denoted by R_+; similarly, the space of elements F_- will be denoted by R_- $(=\overline{R}_+)$. (It is clear that R_+ $[R_-]$ can be regarded as the space of Fourier transforms of functions in $L^2(-\infty, \infty)$ which are 0 on the left [right] half-line. Since orthogonality is preserved under Fourier transforms it follows in particular that $R_+\perp R_-$.)

If $f \in -L^2$ $(0, \infty)$ then

$$(Tf)^\wedge(\lambda) = \int_0^{\infty} e^{-i\lambda t}\left[\int_0^t k(s-t)f(s)ds\right]dt$$

and hence, on inverting the order of integration, $(Tf)^\wedge(\lambda)=\overline{K}_+(\lambda)\hat{f}(\lambda)$, where $K_+(\lambda)$ is defined by

$$K_+(\lambda) = \int_0^{\infty} e^{i\lambda t}k(t)dt. \tag{10}$$

(It may be noted that $\int_0^t k(s-t)f(s)ds$ is the convolution of $\bar{k}$ and f on $0 \leqq t < \infty$.) More generally, an iteration shows that

$$(11) \qquad (T^n f)\hat{}(\lambda) = \overline{K}_+^n \hat{f}(\lambda), \qquad n = 0, 1, 2, \cdots.$$

Let $g \perp \mathfrak{N}_{B^*B}$, so that $Bg = 0$, that is $\int_0^\infty k(t+s)g(s)ds$ is the 0 element of $\mathfrak{H} = L^2(0, \infty)$. Since, by (2), $k(t+s)$ belongs to $L^2(0, \infty)$ for any fixed t, it follows that the last integral is a continuous function of t on $0 \leqq t < \infty$ and that

$$(12) \qquad m_c \in \mathfrak{N}_{B^*B}, \quad \text{where} \quad m_c(t) = \bar{k}(t+c) \quad \text{and} \quad c \geqq 0.$$

It is readily verified that

$$(13) \qquad \hat{m}_c(\lambda) = e^{i\lambda c} \int_c^\infty e^{-i\lambda t} \bar{k}(t) dt.$$

Also, in view of the definition of $K(\lambda)$ in (6), one has

$$(14) \qquad K(\lambda) = K_+(\lambda) + \overline{K}_+(\lambda),$$

where $K_+(\lambda)$ is defined in (10).

In order to prove that $\mathfrak{H}_a(A) = \mathfrak{H}(= L^2(0, \infty))$, it is sufficient, as noted above, to prove that $\mathfrak{M}_T = \mathfrak{H}$. Now, if $\mathfrak{M}_T \neq \mathfrak{H}$, then there exists a function $q \in \mathfrak{H}$ such that $q \neq 0$ and $q \perp \mathfrak{M}_T$. Let $Q = Q(\lambda)$ denote the Fourier transform of q, so that

$$(15) \qquad Q(\lambda) = \int_0^\infty e^{-i\lambda t} q(t) dt \qquad (\in R_-).$$

In view of (12), it follows from the relation $q \perp M_T$ and the fact that orthogonality is preserved under Fourier transforms that

$$(16) \quad Q \perp (T^n f)\hat{}(\lambda),\ n = 0, 1, 2, \cdots, \quad \text{where} \quad f(t) = m_c(t),\ c \geqq 0.$$

Thus, by (11) and (13), $Q \perp \overline{K}_+^n e^{i\lambda c} \int_c^\infty e^{-i\lambda t} \bar{k}(t) dt$, for $c \geqq 0$, that is,

$$(17) \quad Q \perp \overline{K}_+^{n+1} e^{i\lambda c} - \overline{K}_+^n e^{i\lambda c} \int_0^c e^{-i\lambda t} \bar{k}(t) dt \quad (n = 0, 1, 2, \cdots).$$

Since Q and $e^{i\lambda c} \int_0^c e^{-i\lambda t} \bar{k}(t) dt$ belong to R_- and R_+ respectively, it follows from (17) for $n = 0$ that $Q \perp \overline{K}_+ e^{i\lambda c}$ (therefore $Q \perp \overline{K}_+ R_+$) and hence, by induction, that

$$(18) \qquad Q \perp e^{i\lambda c} \overline{K}_+^n, \quad n = 1, 2, \cdots, \quad c \geqq 0.$$

Relations (14) and (18) and the fact that $Q \perp R_+$ imply that $Q \perp K^n(\lambda) R_+$ for $n = 0, 1, 2, \cdots$, that is,

$$\int_{-\infty}^{\infty} K^n(\lambda)F_+(\lambda)\bar{Q}(\lambda)d\lambda = 0 \qquad \text{for } n = 0, 1, 2, \cdots, \tag{19}$$

where $K(\lambda)$ is given in (6) and $F_+(\lambda)$ is an arbitrary element of R_+.

Since $k(t) \in L^1(-\infty, \infty)$, the function $K(\lambda)$ is continuous and satisfies

$$K(\lambda) \to 0 \quad \text{as} \quad |\lambda| \to \infty. \tag{20}$$

Also, by (2), $K(\lambda)$ is real. Let $f(K)$ denote the characteristic function of the K-set: $|K| \geqq 1/n$, where n is a positive integer. It follows from (19), Weierstrass' approximation theorem, and the fact that $F_+\bar{Q}$ is in $L^1(-\infty, \infty)$, that $\int_{-\infty}^{\infty} f(K(\lambda))F_+(\lambda)\bar{Q}(\lambda)d\lambda = 0$ and hence

$$\int_{E_n} F_+(\lambda)\bar{Q}(\lambda)d\lambda = 0, \tag{21}$$

where $E_n = \{\lambda: |K(\lambda)| \geqq 1/n\}$. Since, for $c \geqq 0$, $e^{i\lambda c}\bar{Q}(\lambda)$ is in R_+, one can choose F_+ in (21) to be $e^{i\lambda c}\bar{Q}$ and so $\int_{E_n} e^{i\lambda c}\bar{Q}^2 d\lambda = 0$. In view of (20),

$$E_n \text{ is a bounded set.} \tag{22}$$

Since $\bar{Q}^2$ is in $L^1(-\infty, \infty)$, one can therefore differentiate under the last integral with respect to c and let $c \to 0+$ to obtain $\int_{E_n} \lambda^m \bar{Q}^2 d\lambda = 0$ ($m = 0, 1, 2, \cdots; n = 1, 2, \cdots$). Again, using (22) and Weierstrass' theorem, one concludes that $Q(\lambda) = 0$ a.e. on E_n. In virtue of (6), the set $\bigcup_{n=1}^{\infty} E_n$ differs from $(-\infty, \infty)$ by a set of measure zero and hence $Q(\lambda) = 0$ a.e. on $(-\infty, \infty)$. This implies that $q(t) = 0$ a.e. on $0 \leqq t < \infty$, a contradiction. Thus $\mathfrak{M}_T = \mathfrak{H}$ and so $\mathfrak{H}_a(A) = \mathfrak{H}$ as was to be proved.

References

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Purdue University