GLOBAL SOLUTIONS OF CERTAIN HYPERBOLIC SYSTEMS OF QUASI-LINEAR EQUATIONS

BY J. L. JOHNSON[1] AND J. A. SMOLLER[2]

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We consider systems of the form

$$(1) \qquad u_t + f(v)_x = 0, \qquad v_t + g(u)_x = 0,$$

with initial data $(v(0, x), u(0, x)) = (v_0(x), u_0(x))$. Here u and v are functions of t and x, $t \geqq 0$, $-\infty < x < \infty$, and f and g are C^2 functions of a single real variable. We assume that the system (1) is hyperbolic and genuinely nonlinear in the sense of Lax [4].

THEOREM 1. *For each point (v_0, u_0) in the $(v-u)$-plane, there exist two smooth curves $u = w(v) = w(v, v_0, u_0)$ and $u = s(v) = s(v, v_0, u_0)$, passing through (v_0, u_0) defined for all $v \geqq v_0$ with the properties that $w'(v) > 0$, $s'(v) < 0$ and each point $(v, w(v))$ satisfies the Lax conditions for backward rarefaction waves* [4], *while each point $(v, s(v))$ satisfies the Lax conditions for forward shock waves* [4].

In other words, the Riemann problem for (1) with initial data

$$\begin{aligned} (v_0(x), u_0(x)) &= (v_0, u_0), \qquad & x < 0, \\ &= (v_1, w(v_1)), \qquad & x > 0 \end{aligned}$$

where $v_1 > v_0$, can be solved by two constant states (v_0, u_0) and $(v_1, w(v_1))$ separated by a backward rarefaction wave. Similarly the Riemann problem for (1) with initial data

$$\begin{aligned} (v_0(x), u_0(x)) &= (v_0, u_0), \qquad & x < 0, \\ &= (v_1, s(v_1)), \qquad & x > 0 \end{aligned}$$

where $v_1 > v_0$ can be solved by two constant states (v_0, u_0) and $(v_1, s(v_1))$ separated by a forward shock wave.

Fix a point (v_0, u_0) in $(v-u)$-space and let

$$C(v_0, u_0) = \{(v, u)\colon v \geqq v_0, \quad s(v, v_0, u_0) \leqq u \leqq w(v, v_0, u_0)\}$$

THEOREM 2. *If $(v_1, u_1) \in C(v_0, u_0)$, then $C(v_1, u_1) \subset C(v_0, u_0)$.*

One consequence of Theorem 2 is that the interaction of two forward shocks produces a forward shock and a backward rarefaction

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wave. (A similar result is valid for the interaction of two backward shocks.) In [3] this consequence is part of the hypothesis.

THEOREM 3. *Let one of the functions* $v_0(x)$ *and* $u_0(x)$ *be bounded and let them have the property that if* $x_1<x_2$, $(v_2, u_2)\in C(v_1, u_1)$, *where* $(v_i, u_i) = (v_0(x_i), u_0(x_i))$, $i=1, 2$. *Then there exists a global solution, defined in* $t\geqq 0$, *of* (1) *with the initial data* $(v(0, x), u(0, x))=(v_0(x), u_0(x))$.

The condition on the initial data can be restated as follows. If $x_1<x_2$, then the Riemann problem for (1) with data

$$\begin{aligned}(v_0(x), u_0(x)) &= (v_1, u_1), \qquad x < 0,\\ &= (v_2, u_2), \qquad x > 0\end{aligned}$$

is solvable by a backward rarefaction wave and a forward shock.

Similar theorems can be proved for backward shocks and forward rarefaction waves.

Our methods are extensions of those in [5] where the case $g''(u)=0$ is considered. We obtain the solution as a limit of a sequence of solutions of initial-value problems for (1) with step data. We then show that the approximating solutions are uniformly bounded and have uniformly bounded variation in the sense of Tonelli-Cesari [1], on each compact set in $(t-x)$-space, $t\geqq 0$.

We remark that existence theorems of a somewhat different nature have recently been obtained in [2] and [3], by different methods.

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