GLOBAL SOLUTIONS OF CERTAIN HYPERBOLIC SYSTEMS OF QUASI-LINEAR EQUATIONS

BY J. L. JOHNSON$^1$ AND J. A. SMOLLER$^2$

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We consider systems of the form

\[ u_t + f(v)x = 0, \quad v_t + g(u)x = 0, \]

with initial data \((v(0, x), u(0, x)) = (v_0(x), u_0(x))\). Here \(u\) and \(v\) are functions of \(t\) and \(x\), \(t \geq 0, \quad -\infty < x < \infty\), and \(f\) and \(g\) are \(C^2\) functions of a single real variable. We assume that the system (1) is hyperbolic and genuinely nonlinear in the sense of Lax [4].

**Theorem 1.** For each point \((v_0, u_0)\) in the \((v-u)\)-plane, there exist two smooth curves \(u = w(v) = w(v, v_0, u_0)\) and \(u = s(v) = s(v, v_0, u_0)\), passing through \((v_0, u_0)\) defined for all \(v \geq v_0\) with the properties that \(w'(v) > 0\), \(s'(v) < 0\) and each point \((v, w(v))\) satisfies the Lax conditions for backward rarefaction waves [4], while each point \((v, s(v))\) satisfies the Lax conditions for forward shock waves [4].

In other words, the Riemann problem for (1) with initial data

\[ (v_0(x), u_0(x)) = (v_0, u_0), \quad x < 0, \]
\[ = (v_1, w(v_1)), \quad x > 0 \]

where \(v_1 > v_0\), can be solved by two constant states \((v_0, u_0)\) and \((v_1, w(v_1))\) separated by a backward rarefaction wave. Similarly the Riemann problem for (1) with initial data

\[ (v_0(x), u_0(x)) = (v_0, u_0), \quad x < 0, \]
\[ = (v_1, s(v_1)), \quad x > 0 \]

where \(v_1 > v_0\) can be solved by two constant states \((v_0, u_0)\) and \((v_1, s(v_1))\) separated by a forward shock wave.

Fix a point \((v_0, u_0)\) in \((v-u)\)-space and let

\[ C(v_0, u_0) = \{ (v, u) : v \geq v_0, \quad s(v, v_0, u_0) \leq u \leq w(v, v_0, u_0) \} \]

**Theorem 2.** If \((v_1, u_1) \in C(v_0, u_0)\), then \(C(v_1, u_1) \subset C(v_0, u_0)\).

One consequence of Theorem 2 is that the interaction of two forward shocks produces a forward shock and a backward rarefaction

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wave. (A similar result is valid for the interaction of two backward shocks.) In [3] this consequence is part of the hypothesis.

**Theorem 3.** Let one of the functions \( v_0(x) \) and \( u_0(x) \) be bounded and let them have the property that if \( x_1 < x_2, (v_2, u_2) \in C(v_1, u_1) \), where \((v_i, u_i) = (v_0(x_i), u_0(x_i)), i=1, 2\). Then there exists a global solution, defined in \( t \geq 0 \), of (1) with the initial data \((v(0, x), u(0, x)) = (v_0(x), u_0(x))\).

The condition on the initial data can be restated as follows. If \( x_1 < x_2 \), then the Riemann problem for (1) with data

\[
(v_0(x), u_0(x)) = (v_1, u_1), \quad x < 0,
\]

\[
= (v_2, u_2), \quad x > 0
\]

is solvable by a backward rarefaction wave and a forward shock.

Similar theorems can be proved for backward shocks and forward rarefaction waves.

Our methods are extensions of those in [5] where the case \( g''(u) = 0 \) is considered. We obtain the solution as a limit of a sequence of solutions of initial-value problems for (1) with step data. We then show that the approximating solutions are uniformly bounded and have uniformly bounded variation in the sense of Tonelli-Cesari [1], on each compact set in \((t-x)\)-space, \( t \geq 0 \).

We remark that existence theorems of a somewhat different nature have recently been obtained in [2] and [3], by different methods.

**References**


**The University of Michigan**