A THEOREM ON RANK WITH APPLICATIONS TO MAPPINGS ON SYMMETRY CLASSES OF TENSORS

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1. Results. Let $R$ be a field containing a real closed subfield $R_0$. The main results of this announcement follow.

**Theorem 1.** Let $A_1, A_2, \ldots, A_p$ be $m \times n$ matrices with entries in an infinite subset $\Omega$ of $R$ containing the natural numbers in $R_0$. Let $k$ be a positive integer and assume that the rank of each $A_i$ is at least $k$. Then there exist nonsingular matrices $E$ and $F$ with entries in $\mathbb{Q}$ such that every set of $k$ rows (columns) of $EA_iF$ is linearly independent, $i=1, \ldots, p$.

**Corollary 1.** If the matrices $A_1, \ldots, A_p$ in Theorem 1 each have rank precisely $k$ then every $k$-square subdeterminant of $EA_iF$ is nonzero, $i=1, \ldots, p$.

**Theorem 2.** If $A_1, \ldots, A_p$ are $n$-square complex hermitian matrices all of rank at least $k$ then there exists a nonsingular matrix $E$ such that every set of $k$ rows (columns) of $E^*A_iE$ is linearly independent.

In 1933, J. Williamson [1] gave necessary and sufficient conditions for the compounds of two matrices to be equal. The nontrivial part of his result states the following: if $A$ is a complex matrix of rank $r$ and $r>m$ then $C_m(A) = C_m(B)$ if and only if $A = zB$ where $z^m = 1$. A result closely connected to Theorem 1 and generalizing the Williamson result can be proved. We state our theorem in an invariant setting.

Thus, let $V$ be an $n$-dimensional space over the complex numbers, let $H$ be a subgroup of the symmetric group $S_m, m \leq n$, and let $\chi$ be a complex valued character of degree 1 on $H$. A multilinear function $f(v_1, \ldots, v_m)$ is symmetric with respect to $H$ and $\chi$ if $f(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \chi(\sigma)f(v_1, \ldots, v_m)$ for all $v_1, \ldots, v_m$ in $V$ and all $\sigma \in H$. Let $P$ be a vector space and $f$ a fixed multilinear function symmetric with respect to $H$ and $\chi$, $f: V \times \cdots \times V \to P$, such that for any multilinear function $g, g: V \times \cdots \times V \to U$, also symmetric with respect to $H$ and $\chi$, there exists a linear $h: P \to U$ that makes the following diagram commutative:
Then the pair \( P, f \) is called a *symmetry class of tensors* associated with \( H \) and \( \chi \), e.g., \( H = S_m, \chi = \text{sgn}, P = \Lambda^m V, f(v_1, \ldots, v_m) = v_1 \Lambda \cdots \Lambda v_m \), the usual \( m \)th Grassmann product. If \( T \) is a linear transformation on \( V \) then one defines a linear transformation \( h \) via the diagram (1) with \( U = P, g(v_1, \ldots, v_m) = f(Tv_1, \ldots, Tv_m) \). In this case \( h \) is called the transformation induced by \( T \) and will be denoted here by \( K(T) \).

We have the following generalization of Williamson’s result to an arbitrary symmetry class of tensors as described above. We do not present a proof here but this generalization depends directly on Theorem 1 for the case \( p = 2 \).

**Theorem 3.** If the rank of \( T \) is \( r \) and \( r > m \), then \( K(T) = K(S) \) if and only if \( T = zS \) where \( z^m = 1 \).

**Corollary 2.** If \( V \) is a unitary space, the rank of \( T \) is \( r \), and \( r > m \), then \( T \) is normal if and only if \( K(T) \) is normal.

2. **Proof outline.** We say that a set of \( m \times n \) matrices \( (A_1, \ldots, A_p) \) have property \( R_k \) if there exists a nonsingular \( n \)-square matrix \( F \) such that every set of \( k \) columns of \( A_i F, i = 1, \ldots, p \), is linearly independent: this is abbreviated \( (A_1, \ldots, A_p) \subseteq R_k \). It is clear that if we can prove that any set of \( p \) matrices all of rank at least \( k \) satisfy \( (A_1, \ldots, A_p) \subseteq R_k \) then Theorem 1 will follow. Observe that if \( S_1, \ldots, S_p \) are nonsingular \( m \)-square matrices then

\[
(S_1 A_1, \ldots, S_p A_p) \subseteq R_k
\]

if and only if \( (A_1, \ldots, A_p) \subseteq R_k \).

Now let \( L \) be the \( n \)-square matrix whose \((i, j)\) entry is \( i^j, i, j = 1, \ldots, n \). It is routine to verify that every subdeterminant of every order of \( L \) is nonzero. Next, let \( t_1, \ldots, t_n \) be independent indeterminates over \( \Omega \) and define an \( n \)-square matrix \( L(t_1, \ldots, t_n) \) over \( \Omega[t_1, \ldots, t_n] \) whose \((i, j)\) entry is \( t_i^j, i, j = 1, \ldots, n \). It follows that any specialization of \( t_1, \ldots, t_n \) to nonzero elements of \( \Omega \) pro-
duces a matrix every one of whose subdeterminants is nonzero. According to (2) we can take \((A_1, \cdots, A_p) = (H_1, \cdots, H_p)\) where \(H_i\) is the Hermite normal form of \(A_i\), \(i = 1, \cdots, p\). Next, consider the matrices \(B_i = H_i L(t_1, \cdots, t_n)\) and define the polynomial \(p_i(t_1, \cdots, t_n)\) to be the product of all \(C_{n,k}\) entries in the first row of the \(k\)th compound matrix of \(B_i\), i.e., \(C_k(B_i) = C_k(H_i) C_k(L(t_1, \cdots, t_n))\). The fact that \(A_i\) and hence \(H_i\) has rank at least \(k\) implies that there exists a specialization of \(p_i\) which is not zero. Hence the polynomial

\[
P(t_1, \cdots, t_n) = \prod_{i=1}^{p} p_i(t_1, \cdots, t_n)
\]

is not zero. It follows from a standard theorem on polynomials that there exist nonzero elements \(\xi_1, \cdots, \xi_n\) in \(\Omega\) for which \(P(\xi_1, \cdots, \xi_n) \neq 0\). In other words, there exist nonzero \(\xi_1, \cdots, \xi_n\) in \(\Omega\) for which every entry in the first row of each of \(C_k(H_i L(\xi_1, \cdots, \xi_n))\) is nonzero, \(i = 1, \cdots, p\). This means that every set of \(k\) columns of each of \(H_i L(\xi_1, \cdots, \xi_n)\) is linearly independent and proves the result.

The rest of the results announced above follow from Theorem 1.

**Reference**


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