THE WEIERSTRASS TRANSFORMATION OF CERTAIN GENERALIZED FUNCTIONS\textsuperscript{1}

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The convolution transformation

\[ F(s) = \int_{-\infty}^{\infty} f(t)G(s - t)dt \]

considered by Hirschman and Widder [1] has a kernel \( G \) of the form

\[ G(\tau) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp(\sigma\tau)}{E(\sigma)} d\sigma \]

where

\[ E(\sigma) = \exp(-c\sigma^2 + b\sigma) \prod_{r=1}^{\infty} \left( 1 - \frac{\sigma}{a_r} \right) \exp(\sigma/a_r), \]

the \( c, b, \) and \( a_r \) are real, \( c \geq 0, a_r \neq 0, |a_r| \to \infty, \) and \( \sum a_r^{-2} < \infty. \) In a previous note [2] we extended the convolution transformation to a certain class of generalized functions in the case where \( c = 0 \) in (3). On the other hand, if we substitute the previously neglected factor \( \exp(-c\sigma^2) \) in place of \( E(\sigma) \) in (2) and normalized by setting \( c = 1, \) we obtain

\[ G(\tau) = k(\tau, 1) \]

where

\[ k(\tau, t) = \exp(-\tau^2/4t)/(4\pi t)^{1/2}, \quad -\infty < \tau < \infty, \quad 0 < t < \infty. \]

The convolution transformation (1) then becomes the Weierstrass transformation [1; Chapter VIII]:

\[ F(s) = \frac{1}{(4\pi)^{1/2}} \int_{-\infty}^{\infty} f(\tau) \exp[-(s - \tau)^2/4] d\tau. \]

Here, we round out our previous results by extending (5) to certain generalized functions.

Let \( a \) and \( b \) be fixed real numbers with \( a < b. \) Let \( \rho_{a,b}(\tau) \) be a posi-

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tive (never zero) function on \(-\infty < \tau < \infty\) which is smooth (i.e., has continuous derivatives of all orders) and satisfies

\[
\rho_{\alpha, \beta}(\tau) = \exp(-\beta \tau/2), \quad 1 < \tau < \infty;
\]
\[
= \exp(-\alpha \tau/2), \quad -\infty < \tau < -1.
\]

\(\mathcal{W}_{\alpha, \beta}\) is the linear space of all complex-valued smooth functions \(\phi(\tau)\) on \(-\infty < \tau < \infty\) such that for each \(n = 0, 1, 2, \cdots\)

\[
\gamma_n(\phi) = \max_{0 \leq p \leq n} \sup_{-\infty < \tau < \infty} \left| \exp(\tau^2/4)\rho_{\alpha, \beta}(\tau)\phi^{(n)}(\tau) \right| < \infty.
\]

We assign to \(\mathcal{W}_{\alpha, \beta}\) the topology generated by the set of seminorms \(\{\gamma_n\}\), thereby making \(\mathcal{W}_{\alpha, \beta}\) a sequentially complete countably normed space [3, p. 6]. The dual \(\mathcal{W}'_{\alpha, \beta}\) of \(\mathcal{W}_{\alpha, \beta}\) is a space of generalized functions, whose restrictions to Schwartz's space \(\mathcal{D}\) of smooth functions of compact support are Schwartz distributions [4]. If \(a < \text{Re } s < b\), then \(k(s - \tau, 1)\) as a function of \(\tau\) is in \(\mathcal{W}_{\alpha, \beta}\). Moreover, if \(c \leq a\) and \(b \leq d\), then \(\mathcal{W}_{\alpha, \beta}\) is a subset of \(\mathcal{W}_{c, d}\), and the topology of \(\mathcal{W}_{\alpha, \beta}\) is stronger than that induced on it by \(\mathcal{W}_{c, d}\). Consequently, the restriction of \(f \in \mathcal{W}'_{c, d}\) to \(\mathcal{W}_{\alpha, \beta}\) is in \(\mathcal{W}'_{\alpha, \beta}\). In view of these facts, we can define a generalized Weierstrass transformation as follows:

Let \(f\) be a member of \(\mathcal{W}'_{\alpha, \beta}\) for some \(a\) and \(b\) \((a < b)\). Let \(\sigma_1\) be the infimum of all \(a\) and \(\sigma_2\) the supremum of all \(b\) for which \(f \in \mathcal{W}'_{\alpha, \beta}\). Then, the Weierstrass transform of the generalized function \(f\) is defined by

\[
F(s) = \langle f(\tau), k(s - \tau, 1) \rangle, \quad \sigma_1 < \text{Re } s < \sigma_2.
\]

This has a sense as the application of \(f \in \mathcal{W}'_{\alpha, \beta}\) to \(k(s - \tau, 1) \in \mathcal{W}_{\alpha, \beta}\), where for each given \(s\) we choose \(a\) and \(b\) such that \(\sigma_1 < a < \text{Re } s < b < \sigma_2\).

**Theorem 1.** \(F(s)\) is an analytic function on the strip \(\sigma_1 < \text{Re } s < \sigma_2\), and for each \(n = 1, 2, 3, \cdots\)

\[
D_s^n F(s) = \langle f(\tau), D_s^n k(s - \tau, 1) \rangle, \quad D_s = \partial/\partial s.
\]

Moreover, on any closed substrip \(a \leq \text{Re } s = \sigma \leq b (\sigma_1 < a < b < \sigma_2)\),

\[
|F(\sigma + i\omega)| \leq \exp(\omega^2/4)B(|\omega|)
\]

where \(B\) is a polynomial which depends on \(f\) and the choice of the substrip. These conditions are also sufficient in order for \(F(s)\) to be a Weierstrass transform according to (6).

The proof of this theorem is similar to that of [2; Theorem 1].
The next theorem extends the Hirschman-Widder inversion formula [1, p. 191] to our generalized transformation.

**Theorem 2.** Let \( \sigma \) be any fixed real number such that \( \sigma_1 < \sigma < \sigma_2 \). Then, in the sense of weak convergence in the space \( \mathcal{D}' \) of Schwartz distributions,

\[
\lim_{t \to 1} \int_{-\infty}^{\infty} k(\omega + i\tau - i\sigma, t) F(\sigma + i\omega) d\omega = f(\tau).
\]

This is proven by justifying the steps in the following formal manipulations. For \( \phi \in \mathcal{D}, 0 < t < 1, \) and \( \sigma_1 < \sigma < \sigma_2, \)

\[
\left\langle \int_{-\infty}^{\infty} k(\omega + ix - i\sigma, t) F(\sigma + i\omega) d\omega, \phi(x) \right\rangle
\]

\[
= \left\langle \int_{-\infty}^{\infty} k(\omega + ix - i\sigma, t) (f(\tau), k(\sigma + i\omega - \tau, 1)) d\omega, \phi(x) \right\rangle
\]

\[
= \left\langle f(\tau), \int_{-\infty}^{\infty} k(\omega + ix - i\sigma, t) k(\sigma + i\omega - \tau, 1) d\omega \right\rangle, \phi(x)\right\rangle
\]

\[
= \langle \phi(x), \langle f(\tau), k(x - \tau, 1 - t) \rangle \rangle
\]

\[
= \langle f(\tau), \phi(x), k(x - \tau, 1 - t) \rangle \rangle
\]

\[
\to \langle f(\tau), \phi(\tau) \rangle, \quad t \to 1 - .
\]

By combining these results with those of [2], we can extend the convolution transformation (1), wherein \( G \) is given by (2), to the space \( \mathcal{L}_{\mathcal{D}}' \) of generalized functions defined in [2]; we also obtain an inversion formula for it.

**References**


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