THE WEIERSTRASS TRANSFORMATION OF CERTAIN GENERALIZED FUNCTIONS[1]

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The convolution transformation

$$F(s) = \int_{-\infty}^{\infty} f(t)G(s-t)dt \tag{1}$$

considered by Hirschman and Widder [1] has a kernel G of the form

$$G(\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp(z\tau)}{E(z)}\,dz \tag{2}$$

where

$$E(z) = \exp(-cz^2 + bz)\prod_{\nu=1}^{\infty}\left(1 - \frac{z}{a_\nu}\right)\exp(z/a_\nu), \tag{3}$$

the c, b, and a_ν are real, $c \geqq 0$, $a_\nu \neq 0$, $|a_\nu| \to \infty$, and $\sum a_\nu^{-2} < \infty$. In a previous note [2] we extended the convolution transformation to a certain class of generalized functions in the case where $c=0$ in (3). On the other hand, if we substitute the previously neglected factor $\exp(-cz^2)$ in place of $E(z)$ in (2) and normalized by setting $c=1$, we obtain

$$G(\tau) = k(\tau, 1) \tag{4}$$

where

$$k(\tau, t) = \exp(-\tau^2/4t)/(4\pi t)^{1/2}, \quad -\infty < \tau < \infty, \quad 0 < t < \infty.$$

The convolution transformation (1) then becomes the Weierstrass transformation [1; Chapter VIII]:

$$F(s) = \frac{1}{(4\pi)^{1/2}}\int_{-\infty}^{\infty} f(\tau)\exp[-(s-\tau)^2/4]d\tau. \tag{5}$$

Here, we round out our previous results by extending (5) to certain generalized functions.

Let a and b be fixed real numbers with $a < b$. Let $\rho_{a,b}(\tau)$ be a posi-

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tive (never zero) function on $-\infty<\tau<\infty$ which is smooth (i.e., has continuous derivatives of all orders) and satisfies

$$\begin{aligned}\rho_{a,b}(\tau) &= \exp(-b\tau/2), \quad 1<\tau<\infty,\\ &= \exp(-a\tau/2), \quad -\infty<\tau<-1.\end{aligned}$$

$\mathcal{W}_{a,b}$ is the linear space of all complex-valued smooth functions $\phi(\tau)$ on $-\infty<\tau<\infty$ such that for each $n=0, 1, 2, \cdots$

$$\gamma_n(\phi) = \max_{0\leqq p\leqq n} \sup_{-\infty<\tau<\infty} \left| \exp(\tau^2/4)\rho_{a,b}(\tau)\phi^{(n)}(\tau) \right| < \infty.$$

We assign to $\mathcal{W}_{a,b}$ the topology generated by the set of seminorms $\{\gamma_n\}$, thereby making $\mathcal{W}_{a,b}$ a sequentially complete countably normed space [3, p. 6]. The dual $\mathcal{W}'_{a,b}$ of $\mathcal{W}_{a,b}$ is a space of generalized functions, whose restrictions to Schwartz's space $\mathfrak{D}$ of smooth functions of compact support are Schwartz distributions [4]. If $a<\operatorname{Re} s<b$, then $k(s-\tau, 1)$ as a function of τ is in $\mathcal{W}_{a,b}$. Moreover, if $c\leqq a$ and $b\leqq d$, then $\mathcal{W}_{a,b}$ is a subset of $\mathcal{W}_{c,d}$, and the topology of $\mathcal{W}_{a,b}$ is stronger than that induced on it by $\mathcal{W}_{c,d}$. Consequently, the restriction of $f\in\mathcal{W}'_{c,d}$ to $\mathcal{W}_{a,b}$ is in $\mathcal{W}'_{a,Q}$. In view of these facts, we can define a generalized Weierstrass transformation as follows:

Let f be a member of $\mathcal{W}'_{a,b}$ for some a and b $(a<b)$. Let σ_1 be the infimum of all a and σ_2 the supremum of all b for which $f\in\mathcal{W}'_{a,b}$. Then, the Weierstrass transform of the generalized function f is defined by

$$F(s) = \langle f(\tau), k(s-\tau, 1)\rangle, \quad \sigma_1<\operatorname{Re} s<\sigma_2. \tag{6}$$

This has a sense as the application of $f\in\mathcal{W}'_{a,b}$ to $k(s-\tau, 1)\in\mathcal{W}_{a,b}$, where for each given s we choose a and b such that $\sigma_1<a<\operatorname{Re} s<b<\sigma_2$.

THEOREM 1. *$F(s)$ is an analytic function on the strip $\sigma_1<\operatorname{Re} s<\sigma_2$, and for each $n=1, 2, 3, \cdots$*

$$D_s^n F(s) = \langle f(\tau), D_s^n k(s-\tau, 1)\rangle, \quad D_s = \partial/\partial s. \tag{7}$$

Moreover, on any closed substrip $a\leqq\operatorname{Re} s=\sigma\leqq b(\sigma_1<a<b<\sigma_2)$,

$$\left| F(\sigma+i\omega)\right| \leqq \exp(\omega^2/4)B(|\omega|) \tag{8}$$

where B is a polynomial which depends on f and the choice of the substrip. These conditions are also sufficient in order for $F(s)$ to be a Weierstrass transform according to (6).

The proof of this theorem is similar to that of [2; Theorem 1].

The next theorem extends the Hirschman-Widder inversion formula [1, p. 191] to our generalized transformation.

THEOREM 2. *Let σ be any fixed real number such that $\sigma_1 < \sigma < \sigma_2$. Then, in the sense of weak convergence in the space $\mathfrak{D}'$ of Schwartz distributions,*

$$\lim_{t \to 1-} \int_{-\infty}^{\infty} k(\omega + i\tau - i\sigma, t) F(\sigma + i\omega) d\omega = f(\tau).$$

This is proven by justifying the steps in the following formal manipulations. For $\phi \in \mathfrak{D}$, $0 < t < 1$, and $\sigma_1 < a < \sigma < b < \sigma_2$,

$$\begin{aligned}
&\left\langle \int_{-\infty}^{\infty} k(\omega + ix - i\sigma, t) F(\sigma + i\omega) d\omega, \phi(x) \right\rangle \\
&= \left\langle \int_{-\infty}^{\infty} k(\omega + ix - i\sigma, t) \langle f(\tau), k(\sigma + i\omega - \tau, 1)\rangle d\omega, \phi(x) \right\rangle \\
&= \left\langle \left\langle f(\tau), \int_{-\infty}^{\infty} k(\omega + ix - i\sigma, t) k(\sigma + i\omega - \tau, 1) d\omega \right\rangle, \phi(x) \right\rangle \\
&= \langle \phi(x), \langle f(\tau), k(x - \tau, 1 - t)\rangle\rangle \\
&= \langle f(\tau), \langle \phi(x), k(x - \tau, 1 - t)\rangle\rangle \\
&\to \langle f(\tau), \phi(\tau)\rangle, \quad t \to 1- .
\end{aligned}$$

By combining these results with those of [2], we can extend the convolution transformation (1), wherein G is given by (2), to the space $\mathfrak{L}'_{c,d}$ of generalized functions defined in [2]; we also obtain an inversion formula for it.

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