SYMmetry in NonSelfAdjoint Sturm-Liouville Systems

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Suppose that \( a < b \) and \( C \) is the inner product space of all continuous real-valued functions on \([a, b]\) such that \( ||f|| = (\int_a^b \phi^2)^{1/2} \) if \( f \) is in \( C \). Denote by each of \( p \) and \( q \) a member of \( C \) such that \( \phi(x) > 0 \) for all \( x \) in \([a, b]\). Denote by each of \( W \) and \( Q \) a real \( 2 \times 2 \) matrix and denote by \( C' \) the subspace of \( C \) consisting of all \( f \) in \( C \) such that \( (pf')' - qf \) is in \( C \)

\[
W \begin{bmatrix} f'(a) \\ \phi(a)f'(a) \end{bmatrix} + Q \begin{bmatrix} f'(b) \\ \phi(b)f'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Denote by \( L \) the transformation from \( C' \) to \( C \) such that if \( f \) is in \( C' \), then \( Lf = (pf')' - qf \). Assume for the remainder of this note that \( L \) has an inverse \( T \). The purpose of this note is to point out that if \( T \neq T^* \), it is nevertheless true that \( T \) is very closely related to a symmetric operator. Specifically \( T \) is a dilation (via the two-dimensional space of solutions to the homogeneous equation) of a symmetric operator. This fact permits an analysis of \( T \) using the theory of completely continuous symmetric operators. This suggests a worthwhile alternative to the approach taken in [1, Chapter 12], in which the general theory of completely continuous operators is used.

Denote by \( S' \) the subspace of \( C \) consisting of all \( f \) so that \((pf')' - qf = 0\) and denote by \( S \) the orthogonal complement in \( C \) of \( S' \). Denote by \( P \) the orthogonal projection of \( C \) onto \( S' \).

**Theorem 1.** If \( T \neq T^* \), then \( Tg = T^*g \) if and only if \( g \) is in \( S \).

**Theorem 2.** If \( V \) is the restriction of \((I - P)T\) to \( S \), then \( V^* = V \).

**Indication of Proof of Theorem 1.** From [2] one has that if \( g \) is in \( C \), then the member \( f \) of \( C' \) so that \( Lf = g \) is such that

\[
\begin{bmatrix} f(t) \\ \phi(t)f'(t) \end{bmatrix} = \int_a^b \begin{bmatrix} K_{11}(t, j)K_{12}(t, j) \\ K_{21}(t, j)K_{22}(t, j) \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}
\]

for all \( t \) in \([a, b]\) \( j(x) = x \) if \( x \) is in \([a, b]\)) where

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\[ \begin{bmatrix} K_{11}(t, u) & K_{12}(t, u) \\ K_{21}(t, u) & K_{22}(t, u) \end{bmatrix} = K(t, u) \]

\[ = \begin{cases} M(t, a)[W + QM(b, a)]^{-1}WM(a, u) & \text{if } a \leq u \leq t, \\ -M(t, a)[W + QM(b, a)]^{-1}QM(b, a)M(a, u) & \text{if } t < u \leq b \end{cases} \]

and \( M \) is such that

\[ M(t, u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \int_u^t \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix} M(j, u) \quad \text{for all } t, u \text{ in } [a, b]. \]

\( QM(b, a) \) by \( Z \) and \( \det [W+Z] \) by \( \Delta \). Straightforward computation gives that

\[ \Delta K_{12}(t, u) = \begin{cases} A(t, a)[\det W + (\hat{Z}W)_{11}]B(a, u) + B(t, a)(\hat{Z}W)_{21}B(a, u) \\ + A(t, a)(\hat{Z}W)_{12}D(a, u) + B(t, a)[\det W + (\hat{Z}W)_{22}]D(a, u) & \text{if } a \leq u \leq t \\ -\{ A(t, a)[\det Z + (\hat{W}Z)_{11}]B(a, u) + B(t, a)(\hat{W}Z)_{21}B(a, u) \\ + A(t, a)(\hat{W}Z)_{12}D(a, u) + B(t, a)[\det Z + (\hat{W}Z)_{22}]D(a, u) & \text{if } t < u \leq b \end{cases} \]

since \( A(x, y) = D(y, x), B(x, y) = -B(y, x) \) and \( C(x, y) = -C(y, x) \) if \( x, y \) are in \([a, b]\).

From this it follows that

\[ K_{12}(y, x) - K_{12}(x, y) = (\det W - \det Q)B(y, x)/\Delta \]

for all \( x, y \) in \([a, b]\). Noting that if \( g \) is in \( C \), then the member \( f \) of \( C' \) so that \( Lf = g \) is given by \( f(t) = \int_a^t K_{12}(t, j)g \) for all \( t \) in \([a, b]\), one sees that \( (Tg)(t) = \int_a^t K_{12}(t, j)g \) for all \( t \) in \([a, b]\) and \( g \) in \( C \). Hence if \( g \) is in \( S \) and \( t \) is in \([a, b]\), \( (T^*g)(t) = \int_a^t K_{12}(j, t)g \) and \( (Tg)(t) - (T^*g)(t) = \Delta^{-1}(\det W - \det Q)\int_a^t B(t, j)g \). Hence if \( g \) is in \( S \) and \( t \) is
in \([a, b]\), \((Tg)(t) - (T^*g)(t) = 0\) since \(B(t, j)\) is in \(S'\) inasmuch as \(B(j, t) = -A(j, a)B(t, a) + B(j, a)A(t, a)\).

Suppose \(T \neq T^*\). Then \(\det W = \det Q \neq 0\). Hence if \(g\) is in \(C\) and \(Tg = T^*g\), then \(0 = \int_a^b B(t, j)g = -A(t, a) \int_a^b B(j, a)g + B(t, a) \int_a^b A(j, a)g\) for all \(t\) in \([a, b]\) and hence \(\int_a^b B(j, a)g = 0 = \int_a^b A(j, a)g\). Therefore \(g\) is in \(S\).

**Proof of Theorem 2.** If each of \(h\) and \(g\) is in \(S\), \((Vh, g) = ((I - P)Th, g) = (Th, (I - P)g) = (h, T^*g) = (h, Tg) = ((I - P)h, Tg) = (h, (I - P)Tg) = (h, Vg)\) and so \(V^* = V\).

**References**


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