SYMMETRY IN NONSELFADJOINT STURM-LIOUVILLE SYSTEMS

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Suppose that $a<b$ and C is the inner product space of all continuous real-valued functions on $[a, b]$ such that $\|f\| = (\int_a^b f^2)^{1/2}$ if f is in C. Denote by each of p and q a member of C such that $p(x)>0$ for all x in $[a, b]$. Denote by each of W and Q a real 2×2 matrix and denote by C' the subspace of C consisting of all f in C such that $(pf')'-qf$ is in C and

$$W\begin{bmatrix} f'(a) \\ p(a)f'(a) \end{bmatrix} + Q\begin{bmatrix} f'(b) \\ p(b)f'(b) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Denote by L the transformation from C' to C such that if f is in C', then $Lf = (pf')'-qf$. Assume for the remainder of this note that L has an inverse T. The purpose of this note is to point out that if $T\neq T^*$ it is nevertheless true that T is very closely related to a symmetric operator. Specifically T is a dilation (*via* the two-dimensional space of solutions to the homogeneous equation) of a symmetric operator. This fact permits an analysis of T using the theory of completely continuous symmetric operators. This suggests a worthwhile alternative to the approach taken in [1, Chapter 12], in which the general theory of completely continuous operators is used.

Denote by S' the subspace of C consisting of all f so that $(pf')' - qf = 0$ and denote by S the orthogonal complement in C of S'. Denote by P the orthogonal projection of C onto S'.

THEOREM 1. *If* $T\neq T^*$, *then* $Tg = T^*g$ *if and only if* g *is in* S.

THEOREM 2. *If* V *is the restriction of* $(I-P)T$ *to* S, *then* $V^* = V$.

INDICATION OF PROOF OF THEOREM 1. From [2] one has that if g is in C, then the member f of C' so that $Lf=g$ is such that

$$\begin{bmatrix} f(t) \\ p(t)f'(t) \end{bmatrix} = \int_a^b \begin{bmatrix} K_{11}(t,j)K_{12}(t,j) \\ K_{21}(t,j)K_{22}(t,j) \end{bmatrix}\begin{bmatrix} 0 \\ g \end{bmatrix}$$

for all t in $[a, b]$ ($j(x)=x$ if x is in $[a, b]$) where

$$\begin{bmatrix} K_{11}(t, u) K_{12}(t, u) \\ K_{21}(t, u) K_{22}(t, u) \end{bmatrix} = K(t, u)$$
$$= \begin{cases} M(t, a)[W + QM(b, a)]^{-1} W M(a, u) & \text{if } a \leqq u \leqq t, \\ -M(t, a)[W + QM(b, a)]^{-1} Q M(b, a) M(a, u) & \text{if } t < u \leqq b \end{cases}$$

and M is such that

$$M(t, u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \int_u^t \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix} M(j, u) \quad \text{for all } t, u \text{ in } [a, b].$$

$$M \text{ is denoted by} \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

$QM(b, a)$ by Z and det $[W+Z]$ by Δ. Straightforward computation gives that

$$\Delta K_{12}(t, u)$$
$$= \begin{cases} A(t, a)[\det W + (\hat{Z}W)_{11}]B(a, u) + B(t, a)(\hat{Z}W)_{21}B(a, u) \\ \quad + A(t, a)(\hat{Z}W)_{12}D(a, u) + B(t, a)[\det W + (\hat{Z}W)_{22}]D(a, u) \\ \hfill \text{if } a \leqq u \leqq t \\ -\{A(t, a)[\det Z + (\hat{W}Z)_{11}]B(a, u) + B(t, a)(\hat{W}Z)_{21}B(a, u) \\ \quad + A(t, a)(\hat{W}Z)_{12}D(a, u) + B(t, a)[\det Z + (\hat{W}Z)_{22}]D(a, u)\} \\ \hfill \text{if } t < u \leqq b \end{cases}$$
$$= \begin{cases} B(t, u) \det W - A(t, a)B(u, a)(\hat{Z}W)_{11} + A(t, a)A(u, a)(\hat{Z}W)_{12} \\ \quad - B(t, a)B(u, a)(\hat{Z}W)_{21} + B(t, a)A(u, a)(\hat{Z}W)_{22} \\ \hfill \text{if } a \leqq u \leqq t \\ -B(t, u) \det Z + A(t, a)B(u, a)(\hat{W}Z)_{11} - A(t,a)A(u, a)(\hat{W}Z)_{12} \\ \quad + B(t, a)B(u, a)(\hat{W}Z)_{21} - B(t, a)A(u, a)(\hat{W}Z)_{22} \\ \hfill \text{if } t < u \leqq b, \end{cases}$$

since $A(x, y) = D(y, x)$, $B(x, y) = -B(y, x)$ and $C(x, y) = -C(y, x)$ if x, y are in $[a, b]$.

From this it follows that

$$K_{12}(y, x) - K_{12}(x, y) = (\det W - \det Q)B(y, x)/\Delta$$

for all x, y in $[a, b]$. Noting that if g is in C, then the member f of C' so that $Lf = g$ is given by $f(t) = \int_a^b K_{12}(t, j)g$ for all t in $[a, b]$, one sees that $(Tg)(t) = \int_a^b K_{12}(t, j)g$ for all t in $[a, b]$ and g in C. Hence if g is in C and t is in $[a, b]$, $(T^*g)(t) = \int_a^b K_{12}(j, t)g$ and $(Tg)(t) - (T^*g)(t) = \Delta^{-1}(\det W - \det Q) \int_a^b B(t, j)g$. Hence if g is in S and t is in $[a, b]$, $(Tg)(t)-(T^*g)(t)=0$ since $B(t, j)$ is in S' inasmuch as $B(j, t)=-A(j, a)B(t, a)+B(j, a)A(t, a)$.

Suppose $T\neq T^*$. Then $\det W-\det Q\neq 0$. Hence if g is in C and $Tg=T^*g$, then $0=\int_a^b B(t, j)g=-A(t, a)\int_a^b B(j, a)g+B(t, a)\int_a^b A(j, a)g$ for all t in $[a, b]$ and hence $\int_a^b B(j, a)g=0=\int_a^b A(j, a)g$. Therefore g is in S.

Proof of Theorem 2. If each of h and g is in S, $(Vh, g)=((I-P)Th, g)=(Th, (I-P)g)=(h, T^*g)=(h, Tg)=((I-P)h, Tg)=(h, (I-P)Tg)=(h, Vg)$ and so $V^*=V$.

References

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. J. W. Neuberger, *Concerning boundary value problems*, Pacific J. Math. **10** (1960), 1385–1392.

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