SOME REMARKS ON PARALLELIZABLE STEIN MANIFOLDS

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1. The purpose of this note is to collect some simple facts on the parallelizability of analytic submanifolds of the complex number space C^N, which are remarkable because their analogues in the real case fail to be true. Any analytic submanifold of C^N is a Stein manifold. (An analytic submanifold is closed in C^N by definition.) Conversely, every Stein manifold can be embedded in some C^N, i.e. mapped biholomorphically onto an analytic submanifold of C^N. An n-dimensional Stein manifold X is called parallelizable if there exists a holomorphic field of n-frames on X, i.e. n holomorphic vector fields which are linearly independent at every point $x \in X$. (We require throughout this paper that all connected components of a manifold have the same dimension.) By a theorem of Grauert [2], an n-dimensional Stein manifold X is parallelizable if and only if there exists a continuous field of (complex) n-frames on X. We connect the parallelizability with the notion of complete intersection: An n-dimensional analytic submanifold X of C^N is called a complete intersection, if the ideal $I(X)$ of all holomorphic functions on C^N which vanish on X can be generated by $N-n$ elements. This is the case if and only if there exist $N-n$ holomorphic functions $f_1, \cdots, f_{N-n}$ on C^N such that

$$X = \{x \in C^N : f_1(x) = \cdots = f_{N-n}(x) = 0\}$$

and the rank of the functional matrix of $(f_1, \cdots, f_{N-n})$ equals $N-n$ at every point $x \in X$. We shall prove that a Stein manifold is parallelizable if and only if it can be embedded as a complete intersection in some complex number space C^N.

2. The following lemma expresses the duality between the normal and tangent bundle of an analytic submanifold of C^N.

LEMMA. *Let X be an n-dimensional analytic submanifold of C^N.*

(i) *If the normal bundle of X is trivial, then X is parallelizable.*

(ii) *If X is parallelizable and $N \geqq 3n/2$, then the normal bundle of X is trivial.*

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PROOF. We use the following theorem (cf. [3]). Let ξ_1 and ξ_2 be two complex vector bundles of rank r over an n-dimensional CW-complex X and η a trivial vector bundle over X. Suppose that $\xi_1 \oplus \eta$ is isomorphic to $\xi_2 \oplus \eta$ and that $r \geqq n/2$. Then ξ_1 and ξ_2 are isomorphic.

Let τ be the tangent bundle of X, ν the normal bundle of X and θ the (trivial) tangent bundle of C^N, restricted to X. Then we have $\tau \oplus \nu \cong \theta$. In order to prove (i) we apply the cited theorem to the case where $\eta = \nu$, $\xi_1 = \tau$ and ξ_2 is the trivial vector bundle of rank n over X. In order to prove (ii) we set $\eta = \tau$, $\xi_1 = \nu$ and $\xi_2 =$ trivial vector bundle of rank $N-n$ over X, using the fact that a (complex) n-dimensional Stein manifold has the homotopy type of a (real) n-dimensional CW-complex (cf. [5]).

THEOREM. *Let X be an n-dimensional analytic submanifold of C^N.*

(i) *If X is a complete intersection, then it is parallelizable.*

(ii) *If X is parallelizable and $N \geqq (3n/2)+1$, then X is a complete intersection.*

PROOF. (i) Because the normal bundle of a complete intersection is trivial, this follows from part (i) of the lemma.

(ii) Part (ii) of the lemma implies that the normal bundle of X is trivial. Since $N \geqq (3n/2)+1$, it follows from [1, Satz 12], that X is a complete intersection.

COROLLARY 1. *A Stein manifold is parallelizable if and only if it can be embedded as a complete intersection in some complex number space C^N.*

This is true because every n-dimensional Stein manifold can be embedded in C^{2n+1}.

COROLLARY 2. *Every $(N-1)$-dimensional analytic submanifold of C^N is parallelizable.*

Since the second Cousin problem in C^N always has a solution, every such submanifold is a complete intersection.

Note that the analogue of the corollary for the real case is not true as the two-sphere

$$\{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1\}$$

shows. (However it is well known that the complexified two-sphere

$$\{(x, y, z) \in C^3 : x^2 + y^2 + z^2 = 1\}$$

is parallelizable.)

COROLLARY 3. *An $(N-2)$-dimensional analytic submanifold X of C^N is parallelizable if and only if its first Chern class $c_1(X)$ vanishes.*

PROOF. It is clear that all Chern classes of X vanish, if X is parallelizable. Conversely, suppose $c_1(X)=0$. Then it has been proved in [1, Hilfssatz 17, Corollary], that X is a complete intersection. Therefore the corollary follows from part (i) of the theorem.

Using this last argument and part (ii) of the theorem, we get

COROLLARY 4. *An analytic submanifold X of C^N, $N\leqq 7$, is a complete intersection if and only if X is parallelizable.*

Thus the property of being a complete intersection in C^N, $N\leqq 7$, depends only on the submanifold and not on its embedding in C^N. It is not known if this remains true in C^N with $N\geqq 8$.

A theorem of Peterson [6] (cf. also [4]) asserts that if X is an n-dimensional CW-complex such that the only torsion in $H^{2q}(X, Z)$ is relatively prime to $(q-1)!$ for $q=1, 2, \cdots$, and ξ is a complex vector bundle of rank $\geqq n/2$ over X, then ξ is trivial if and only if all its Chern classes vanish. Since for a (complex) n-dimensional Stein manifold X we have $H^p(X, Z)=0$ for $p>n$, we can express the parallelizability of low-dimensional Stein manifolds completely in terms of Chern classes.

PROPOSITION. *A Stein manifold X of dimension $\leqq 5$ is parallelizable if and only if $c_1(X)=c_2(X)=0$.*

3. We want to give an example of a Stein manifold which is not parallelizable. Since every one-dimensional Stein manifold is parallelizable (this follows from the above proposition, but can also be proved by more elementary means), the simplest example is of dimension two. Let X be the following open subset of the 2-dimensional complex projective space:

$$X = \{(x:y:z) \in P_2(C): x^2+y^2+z^2 \neq 0\}.$$

It is easy to check that X is Stein. We claim that X is not parallelizable. The real projective plane $P_2(R)$, which is naturally embedded in X, is a deformation retract of X. A deformation of X to $P_2(R)$ is given by the family of maps $F_t: X\to X$, $1\geqq t\geqq 0$, where

$$F_t(x:y:z) = ((\operatorname{Re} x + it \operatorname{Im} x):(\operatorname{Re} y + it \operatorname{Im} y):(\operatorname{Re} z + it \operatorname{Im} z)).$$

Here the homogeneous coordinates $(x:y:z)$ of a point of X have to be chosen in such a way that $x^2+y^2+z^2>0$. Let τ be the tangent bundle of X. We want to calculate the Stiefel-Whitney classes of τ (regarded as a real vector bundle). It suffices to consider the restriction of τ to $P_2(R)$. This restriction is the complexification of τ_0, the real tangent bundle of $P_2(R)$, hence, as a real bundle, isomorphic to the Whitney sum $\tau_0 \oplus \tau_0$. The total Stiefel-Whitney class of τ_0 is $w(\tau_0) = (1+\alpha)^3$, where $\alpha \in H^1(P_2(R), Z_2)$ is the generator of the cohomology ring $H^*(P_2(R), Z_2)$, (cf. [**3**]). Therefore $w(\tau_0 \oplus \tau_0) = (1+\alpha)^6 = 1+\alpha^2$. Since the second Stiefel-Whitney class is the reduction modulo 2 of the first Chern class, it follows that $c_1(X) = c_1(\tau) = \gamma$, where γ is the nonzero element of $H^2(X, Z) \cong H^2(P_2(R), Z) \cong Z_2$. This shows that X is not parallelizable.

By Corollary 2, X cannot be embedded in C^3. However, the mapping $f: X \to C^4$,

$$f(x: y: z) = (x^2 + y^2 + z^2)^{-1}(xy, xz, yz, x^2 - y^2),$$

maps X biholomorphically onto an analytic submanifold X_1 of C^4. By our theorem, X_1 is not a complete intersection (cf. Stiefel [**8**, Anhang II]).

The Stein manifold X can also be used to show that the following result of Ramspott [**7**] is best possible.

Let Y be an n-dimensional Stein manifold. Then for every $k \leqq (n+1)/2$ there exists a holomorphic field of k-frames on Y.

For $n = 2m$ set $Y = X^m$, for $n = 2m+1$ set $Y = X^m \times C$. We claim that there exists no holomorphic field of k-frames on Y if $k > (n+1)/2$. To prove this it suffices to show that $c_m(Y) \neq 0$. The cohomology ring $H^*(Y, Z_2) \cong H^*((P_2(R))^m, Z_2)$ is isomorphic to $Z_2[\alpha_1, \cdots, \alpha_m]$, where $\alpha_1, \cdots, \alpha_m$ are elements of degree 1 with the only relation $\alpha_i^3 = 0$. From our above calculations on X follows that the total Stiefel-Whitney class of the tangent bundle τ of Y is

$$w(\tau) = (1 + \alpha_1^2) \cdot \ . \ . \ . \ \cdot (1 + \alpha_m^2).$$

In particular $w_{2m}(\tau)$ is the nonzero element of $H^{2m}(Y, Z_2) = Z_2$. Therefore $c_m(Y)$ is the nonzero element of $H^{2m}(Y, Z) = Z_2$. q.e.d.

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INSTITUTE FOR ADVANCED STUDY

AUTOMORPHISM GROUPS OF FINITELY GENERATED NILPOTENT GROUPS

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It is rare for any property of a group G to carry over to its automorphism group. Recently J. Lewin [1] constructed a finitely presented group whose automorphism group is not even finitely generated. Now finitely generated nilpotent groups are finitely presented (see e.g. [2]). So Lewin's example contrasts strikingly with the following.

THEOREM A. *The automorphism group of a finitely generated nilpotent group is finitely presented.*

In a way Theorem A reinforces the commonly held view that the automorphism group of a finitely generated nilpotent group is, from a group-theoretical viewpoint, quite simple. Now Philip Hall [3] has proved that a finitely generated nilpotent group has a faithful representation in $\mathrm{GL}(n, Z)$, the integer unimodular group of degree n. So the following generalization of Hall's theorem might be thought of as another indication of the controlled nature of finitely generated nilpotent groups and their automorphism groups.

THEOREM B. *The holomorph of a finitely generated nilpotent group (i.e. the split extension of the group by its automorphism group) has a faithful representation in* $\mathrm{GL}(n, Z)$ *for some* n.

The proofs of Theorem A and Theorem B use general Lie-theoretic techniques and a result which is of independent interest, namely

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