INVOLUTIONS OF HOMOTOPY SPHERES AND HOMOLOGY 3-SPHERES

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1. **Introduction.** Let Σ^{4n+3} be a homotopy sphere and T: $\Sigma \to \Sigma$ a fixed point free differentiable involution. A characteristic submanifold for T is a smoothly embedded submanifold $W^{4n+2} \subset \Sigma$, such that $W = A \cap TA$, $\Sigma = A \cup TA$, where A is a compact submanifold of Σ, $\partial A = W$. Let i: $W \to A$ be the inclusion, and $K = \ker i_*$, $i_*: H_{2n+1}(W) \to H_{2n+1}(A)$. The symmetric bilinear pairing $K \otimes K \to \mathbf{Z}$ defined by $x \otimes y \to x \cdot T_* y$ is called the quadratic form of T with respect to W, and its signature is denoted by $\sigma(T, \Sigma)$. It is proved in [2] that $\sigma(T, \Sigma)$ does not depend on the characteristic submanifold, and that for $n > 0$, $\sigma(T, \Sigma) = 0$ if and only if Σ contains an invariant smoothly embedded S^{4n+2}. These definitions can be made in the p.l. category and the corresponding properties hold. $\sigma(T, \Sigma)$ can also be defined when Σ is a homology sphere.

It is the purpose of this paper to give examples of involutions with $\sigma(T, \Sigma) \neq 0$.

2. We will make use of the following construction: Let M^n be a smooth manifold, and $T: \partial M \to \partial M$ a smooth involution. Consider another copy M^* of M, and the manifold $M' = M \cup_T M^*$, obtained from the disjoint union of M and M^* by identifying $T(x) \in \partial M$ with $x^* \in \partial M^*$. Then an involution T': $M' \to M'$ can be defined by $T'(x) = x^*$, $T'(x^*) = x$. $T' | \partial M = T$ and T' is fixed point free if and only if T is.

We will denote by U the square matrix with 1's in the nonprincipal diagonal and 0's elsewhere.

Let H be a $2k \times 2k$ integral matrix. We will consider the following conditions on H:

(i) $\det H = \pm 1$.

(ii) There exist $2k \times 2k$ integral matrices P, Q, such that $H = PUP^t - QUQ^t$.

(iii) PQ^t is symmetric.

(iv) PQ^t has even integers in the main diagonal.

3. THEOREM 1. *If H satisfies conditions* (i)–(iv), *then H can be realized as the matrix of the quadratic form of a fixed point free differentiable involution of a homotopy* $(4n+3)$-*sphere*, $n > 0$.

THEOREM 1′. *If H satisfies conditions* (i)–(iii), *then H can be realized as the matrix of the quadratic form of a fixed point free involution of a homology 3-sphere.*

PROOF. Let W^{4n+2} be the connected sum of $2k$ copies of $S^{2n+1} \times S^{2n+1}$, and $\alpha_1, \cdots, \alpha_{2k}, \beta_1, \cdots, \beta_{2k}$ the standard basis of $H_{2n+1}(W)$. The matrix of intersection numbers with respect to this basis is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

There is a fixed point free differentiable involution $T: W \to W$ such that the matrix of $T_*: H_{2n+1}(W) \to H_{2n+1}(W)$ with respect to α_i, β_i is U. This can be seen by considering the antipodal map in S^{4n+3} and exchanging handles in S^{4n+2}, or by applying the construction described in §2 to the connected sum of k copies of $S^{2n+1} \times S^{2n+1}$, with a disc removed, and the antipodal map in its boundary.

Consider the basis

$$\alpha_i' = \sum p_{ij}\alpha_j + \sum q_{ij}\beta_j, \qquad \beta_i' = \sum r_{ij}\alpha_j + \sum s_{ij}\beta_j$$

where $(p_{ij}) = P$, $(q_{ij}) = Q$ are the matrices given by (ii), and $(r_{ij}) = R = H^{-1}QU$, $(s_{ij}) = S = H^{-1}PU$. The matrix of intersection numbers with respect to this new basis is

$$(1) \qquad \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

as a result of conditions (ii) and (iii). This shows also that α_i', β_i' is really a basis. The matrix of T_* with respect to α_i', β_i' is

$$(2) \qquad \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 & U \\ U & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} 0 & H \\ H^{-1} & 0 \end{pmatrix}.$$

A function $\phi: H_{2n+1}(W) \to Z_2$ can be defined [3], such that $\phi(x) = 0$ if and only if framed surgery can be performed on an embedded sphere representing x, and has the property

$$\phi(x + y) = \phi(x) + \phi(y) + x \cdot y.$$

Therefore

$$\begin{aligned} \phi(\alpha_i') &= \phi\left(\sum p_{ij}\alpha_j\right) + \phi\left(\sum q_{ij}\beta_j\right) + \left(\sum p_{ij}\alpha_j\right) \cdot \left(\sum q_{ij}\beta_j\right) \\ &= 0 + 0 + \sum p_{ij}q_{ij} \end{aligned}$$

is the ith diagonal element of PQ^t, which is 0 mod 2 by condition (iv).

Therefore we can do surgery on the elements α_i' to obtain a framed cobordism between W and a homotopy sphere, which must be diffeomorphic to S^{4n+2} since it bounds a π-manifold [3]. Attaching a disc to this sphere, we obtain a differentiable manifold V such that $\partial V=W$, $\pi_1(V)=0$, $\tilde{H}_i(V)=0$, $i\neq 2n+1$, $H_{2n+1}(V)$ is free on $2k$ generators $b_1, \cdots, b_{2k}$ and $i_*: H_{2n+1}(W)\to H_{2n+1}(V)$ is given by $\alpha_i'\to 0$, $\beta_i'\to b_i$.

Applying to V, $T: W\to W$, the construction described in §2, we obtain a manifold $\Sigma^{4n+3}=V\cup_T V^*$ and an involution $T': \Sigma\to\Sigma$. $\pi_1(\Sigma)=0$, and it follows from the Mayer-Vietoris sequence of $(\Sigma; V, V^*)$ that Σ is a homotopy sphere, since the only nontrivial part is

$$\begin{array}{c} 0 \to H_{2n+2}(\Sigma) \to H_{2n+1}(W) \to H_{2n+1}(V) \oplus H_{2n+1}(V^*) \to H_{2n+1}(\Sigma) \to 0 \\ (i_*, i_*T_*) \searrow \qquad \wr\wr \\ H_{2n+1}(V) \oplus H_{2n+1}(V) \end{array}$$

and (i_*, i_*T_*) has, with respect to the bases α_i', β_i' and $(b_i, 0)$, $(0, b_i)$, the matrix

$$\begin{pmatrix} 0 & H \\ I & 0 \end{pmatrix}$$

and is an isomorphism by condition (i).

W is a characteristic submanifold for T', K is the subgroup generated by the α_i', and the matrix $(\alpha_i'\cdot T_*\alpha_j')=H$.

For $n=0$, the matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

can be lifted to an automorphism of $\pi_1(W)$, since it satisfies (1) (see [5, pp. 177–178 and 355–356]), and therefore, by Nielsen's Theorem [7, p. 266], it can be realized by a homeomorphism $h: W\to W$. Therefore we can take V to be the solid torus of genus $2k$, with the involution hTh^{-1} on its boundary, and the rest of the proof follows as before.

4. PROPOSITION. *For all $i\in \mathbf{Z}$ there exists a matrix H satisfying* (i)–(iv), *with $\sigma(H)=8i$.*

We give here a simple proof of the existence of such matrices with $\sigma(H)=32i$. To get all multiples of 8, a more elaborate argument is needed, which uses methods developed in [8].

If H is any symmetric matrix with even integers in the main diagonal and $\det H=\pm 1$, then

$$H' = \begin{pmatrix} H & 0 & 0 & 0 \\ 0 & UHU & 0 & 0 \\ 0 & 0 & H & 0 \\ 0 & 0 & 0 & UHU \end{pmatrix}$$

satisfies (i)–(iv), by writing $H = X + X^t$, and taking

$$P = \begin{pmatrix} 0 & XU & I & 0 \\ UX^t & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & -XU \\ 0 & I & -UX^t & 0 \end{pmatrix}.$$

In this case $PQ^t = 0$, and since the signature of H can be any multiple of 8 [6], the signature of H' can be any multiple of 32.

THEOREM 2. *For all $n > 0$, and all $i \in \mathbf{Z}$, there is a fixed point free differentiable involution $T_i: \Sigma_i^{4n+3} \to \Sigma_i^{4n+3}$ of a homotopy sphere Σ_i^{4n+3}, such that $\sigma(T_i, \Sigma_i) = 8i$.*

THEOREM 2′. *For all $i \in \mathbf{Z}$ there is a fixed point free involution $T_i: \Sigma_i^3 \to \Sigma_i^3$ of a homology 3-sphere Σ_i^3, such that $\sigma(T_i, \Sigma_i^3) = 8i$.*

COROLLARY 1. *For $n > 0$, there exists infinitely many differentiable manifolds, homotopy equivalent to P^{4n+3}, no two of which are p.l. homeomorphic.*[1]

COROLLARY 2. *There exist infinitely many different irreducible 3-manifolds with the same homology as P^3.*

5. Final remarks. If one of the homology spheres Σ_i^3, $i \neq 0$, given by Theorem 2′ is simply connected, then, by a theorem of Livesay [4], it would be a counterexample to the Poincaré conjecture.

If in the examples given by Theorem 2 we take the quotient spaces, we can form a diagram

$$\begin{array}{ccc} \Sigma_i^{4n+3}/T_i & \xrightarrow{f} & P^{4n+3} \\ \cup & & \cup \\ W/T & \xrightarrow{g} & P^{4n+2} \end{array}$$

where f is a homotopy equivalence, f is transverse regular to P^{4n+2} and

[1] C. T. C. Wall announced this result at the International Congress of Mathematicians, Moscow, 1966. Montgomery and Yang have recently obtained another proof of Theorem 2 for $n = 1$.

$W/T=f^{-1}(P^{4n+2})$. Then, if $i\neq 0$, it is impossible to do surgery on W/T inside Σ_i/T_i to make g a homotopy equivalence, although the surgery can be done in abstract. (Compare with [1].)

It can be shown that this situation cannot occur in the case of involutions of homotopy $(4n+1)$-spheres, or in the surgery problems that arise in proving uniqueness of desuspension [2]. Therefore, the method given in this paper cannot be used to realize any of the other obstructions defined in [2].

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