ON THE STRUCTURE OF MAXIMALLY ALMOST Periodic Groups

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1. Introduction. A topological group $G$ is said to be maximally almost periodic if the continuous almost periodic functions separate points in $G$, or equivalently if the continuous finite-dimensional unitary representations of $G$ separate points in $G$. See [4], or [2, §18]. Throughout this note, we use “representation” to mean “continuous finite-dimensional unitary representation”. Our purpose here is to announce some results concerning maximally almost periodic (MAP) groups which are independent of the classical theorem of Freudenthal-Weil which states that a locally compact connected group is MAP if and only if it is the direct product of $R^n$ and a compact group [6, §§30, 31].

The results in this note comprise a portion of the author’s doctoral dissertation. Detailed proofs of these and other results will appear at a later date. The author thanks his thesis advisor, Professor Edwin Hewitt, and Professor Lewis Robertson for all their assistance and encouragement.¹

2. Definitions and notation. Let $K$ be a (Hausdorff but not necessarily locally compact) topological group, $G$ a normal subgroup of $K$ and $T = \{ t(x) : x \in K \}$ be the group of topological automorphisms of $G$ which are restrictions to $G$ of inner automorphisms of $K$. Let $\hat{K}$ (and $\hat{G}$ resp.) be the space of equivalence classes of irreducible representations of $K$ (and $G$ resp.). In an investigation of $\hat{K}$ it is natural to consider the action on $\hat{G}$ induced by $T$. For example, see [1]. Let $U$ be a representation, $U \in \sigma \in \hat{G}$, define $t^*(x)U = U \circ t(x)^{-1}$ and define $t^*(x)\sigma$ to be the equivalence class of $t^*(x)U$. If the set $\{ t^*(x)\sigma : t(x) \in T \}$ is finite, then $\sigma$ is said to be finitely orbited by $T$. Let $F(\hat{G}, T)$ be the set $\{ \sigma \in \hat{G} : \sigma$ is finitely orbited by $T \}$. The von Neumann kernel of a group is the intersection of all kernels of representations of that group.

3. Results.

THEOREM 1. Let $K$, $G$ and $T$ be as above. If $U \in \sigma \in \hat{K}$ and if $y \in G$ are such that $U_y \neq I$, then there exists an element of $F(\hat{G}, T)$ which separates

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y from the identity. In particular, if $K$ is MAP, then $F(\hat{G}, T)$ separates points in $G$.

This is proved by utilizing the uniqueness of the decomposition into a direct sum of irreducible constituents of the restriction of $U$ to $G$; the equivalence classes of these constituents are permuted by the action of $T$.

**Theorem 2.** Let $K$, $G$ and $T$ be as above. Let $\sigma \in F(\hat{G}, T)$ and let $O(\sigma, T)$ be the orbit of $\sigma$ by $T$. Then the mapping $\Sigma$ which sends $x$ onto the restriction of $t^*(x)$ to $O(\sigma, T)$ is well defined and is a continuous homomorphism of $K$ onto a finite group. The kernel of $\Sigma$ contains $G$.

In general the condition that $F(\hat{G}, T)$ separate points in $G$ is not enough to imply that $K$ is MAP even if $K/G$ is assumed to be MAP. However, if $K$ is the semidirect product of $G$ and a topological group $H$, $K = G \oplus_H H$, then we have

**Theorem 3.** Let $K = G \oplus_H H$. Let $H_0$ (and $(G \oplus_H H)_0$ resp.) be the von Neumann kernel of $H$ (and $G \oplus_H H$ resp.). Let $S = \bigcap \{ \ker U : U \in \sigma \in F(\hat{G}, \beta(H)) \}$. Then $(G \oplus_H H)_0 = S \oplus_{H_0} H_0$. In particular, $G \oplus_{H_0} H$ is MAP if and only if $H$ is MAP and $F(\hat{G}, H)$ separates points in $G$.

The connection between the $\beta(H)$ here and the $T$ above follows from the equation $t(e, h)(x, e) = (\beta(h)(x), e)$. See [2, p. 7]. The major difficulty in the proof of this theorem is to show that if $g \in G$ and if $U \in \sigma \in F(\hat{G}, \beta(H))$ are such that $U_g \neq I$, then there exists a representation $V$ of $K$ which separates $(g, e)$ from the identity. A rough sketch follows. Let $\Sigma$ be the homomorphism corresponding to $\sigma$ defined in Theorem 2. Then $\ker \Sigma = G \oplus_M$ and $(G \oplus_H H)/(G \oplus M)$ is a finite group. Let $\mathcal{U}(n)$ be the unitary group of $U$ and use Burnside's theorem [3, p. 276] to know that the set $\{ U_x : x \in G \}$ spans the $n^2$-dimensional Hilbert space of all linear operators on $C^* (C$ is the field of complex numbers). A closed subgroup $\mathfrak{A}$ of $\mathcal{U}(n^2)$, a semidirect product $\mathcal{U}(n) \oplus \mathfrak{A}$ and a continuous homomorphism $\phi$ of $G \oplus M$ into $\mathcal{U}(n) \oplus \mathfrak{A}$ are constructed. Then $\phi(g, e)$ can be separated from the identity by a representation $W$ of the compact group $\mathcal{U}(n) \oplus \mathfrak{A}$ and the desired representation $V$ of $K$ is induced from the representation $W \circ \phi$ of $\ker \Sigma$.

If $G$ is an Abelian group, then we can identify the character group $X$ of $G$ with $\hat{G}$ and with the notation as in 2 above, $F(X, T)$ is a subgroup of $X$.

**Theorem 4.** Let $V$ be a normal subgroup of a topological group $K$. Assume further that $V$ is topologically isomorphic to the additive group
of a finite-dimensional vector space over some locally compact, nondiscrete field of characteristic zero. Let \( C \) be the centralizer of \( V \) in \( K \). Then \( K \) is MAP if and only if \( C \) is MAP and \( K/C \) is a finite group.

We make use of Pontrjagin's classification of locally compact fields \([5, \text{ Satz 22}]\) and the fact that the field of real numbers and the \( p \)-adic number fields are self-dual to show that the finitely orbited characters of \( V \) form a subspace of \( V \), so that \( F(V, T) \) is closed in \( V \). Furthermore, it follows from Theorem 1 that \( F(V, T) \) is dense in \( V \). These facts imply that \( T \) must be finite so that \( C \) must have finite index in \( K \).

Using a \( p \)-series field, a group can be constructed to show that the hypothesis above (that the field have characteristic zero) is essential.

**References**