ON THE STRUCTURE OF MAXIMALLY ALMOST
PERIODIC GROUPS

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1. Introduction. A topological group $G$ is said to be maximally almost
periodic if the continuous almost periodic functions separate
points in $G$, or equivalently if the continuous finite-dimensional uni­
tary representations of $G$ separate points in $G$. See [4], or [2, §18].
Throughout this note, we use “representation” to mean “continuous
finite-dimensional unitary representation”. Our purpose here is to
announce some results concerning maximally almost periodic (MAP)
groups which are independent of the classical theorem of Freudenthal-
Weil which states that a locally compact connected group is MAP
if and only if it is the direct product of $\mathbb{R}^n$ and a compact group
[6, §§30, 31].

The results in this note comprise a portion of the author’s doctoral
dissertation. Detailed proofs of these and other results will appear at a
later date. The author thanks his thesis advisor, Professor Edwin
Hewitt, and Professor Lewis Robertson for all their assistance and
encouragement.\\1

2. Definitions and notation. Let $K$ be a (Hausdorff but not neces­
sarily locally compact) topological group, $G$ a normal subgroup of $K$
and $T = \{t(x): x \in K\}$ be the group of topological automorphisms of
$G$ which are restrictions to $G$ of inner automorphisms of $K$. Let $\hat{K}$
(and $\hat{G}$ resp.) be the space of equivalence classes of irreducible repre­
sentations of $K$ (and $G$ resp.). In an investigation of $\hat{K}$ it is natural to
consider the action on $\hat{G}$ induced by $T$. For example, see [1]. Let $U$
be a representation, $U \subset \sigma \subset \hat{G}$, define $t^*(x)U = U \circ t(x)^{-1}$ and define
$t^*(x)\sigma$ to be the equivalence class of $t^*(x)U$. If the set $\{t^*(x)\sigma:
(t(x) \in T\}$ is finite, then $\sigma$ is said to be finitely orbited by $T$. Let $F(\hat{G}, T)$
be the set $\{\sigma \in \hat{G}: \sigma$ is finitely orbited by $T\}$. The von Neumann kernel
of a group is the intersection of all kernels of representations of that
group.

3. Results.

THEOREM 1. Let $K, G$ and $T$ be as above. If $U \subset \sigma \subset \hat{K}$ and if $y \in G$
are such that $U_y \neq I$, then there exists an element of $F(\hat{G}, T)$ which separates

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sity of Washington.
This is proved by utilizing the uniqueness of the decomposition into a direct sum of irreducible constituents of the restriction of $U$ to $G$; the equivalence classes of these constituents are permuted by the action of $T$.

**Theorem 2.** Let $K$, $G$ and $T$ be as above. Let $\sigma \in F(\hat{G}, T)$ and let $O(\sigma, T)$ be the orbit of $\sigma$ by $T$. Then the mapping $\Sigma$ which sends $x$ onto the restriction of $t^*(x)$ to $O(\sigma, T)$ is well defined and is a continuous homomorphism of $K$ onto a finite group. The kernel of $\Sigma$ contains $G$.

In general the condition that $F(\hat{G}, T)$ separate points in $G$ is not enough to imply that $K$ is MAP even if $K/G$ is assumed to be MAP. However, if $K$ is the semidirect product of $G$ and a topological group $H$, $K = G \rtimes H$, then we have

**Theorem 3.** Let $K = G \rtimes H$. Let $H_0$ (and $(G \rtimes H)_0$ resp.) be the von Neumann kernel of $H$ (and $G \rtimes H$ resp.). Let $S = \cap \{ \ker U : U \in \sigma \in F(\hat{G}, \beta(H)) \}$. Then $(G \rtimes H)_0 = S \rtimes H_0$. In particular, $G \rtimes H$ is MAP if and only if $H$ is MAP and $F(\hat{G}, \beta(H))$ separates points in $G$.

The connection between $\beta(H)$ here and the $T$ above follows from the equation $t(e, h)(x, e) = (\beta(h)(x), e)$. See [2, p. 7]. The major difficulty in the proof of this theorem is to show that if $g \in G$ and if $U \in \sigma \in F(\hat{G}, \beta(H))$ are such that $U_g \neq I$, then there exists a representation $V$ of $K$ which separates $(g, e)$ from the identity. A rough sketch follows. Let $\Sigma$ be the homomorphism corresponding to $\sigma$ defined in Theorem 2. Then $\ker \Sigma = G \rtimes M$ and $(G \rtimes H)/(G \rtimes M)$ is a finite group.

Let $\mathfrak{U}(n)$ be the unitary group of $U$ and use Burnside’s theorem [3, p. 276] to know that the set $\{ U_x : x \in G \}$ spans the $n^2$-dimensional Hilbert space of all linear operators on $C^*$ ($C$ is the field of complex numbers). A closed subgroup $\mathfrak{H}$ of $\mathfrak{U}(n^2)$, a semidirect product $\mathfrak{U}(n) \rtimes \mathfrak{H}$ and a continuous homomorphism $\phi$ of $G \rtimes M$ into $\mathfrak{U}(n) \rtimes \mathfrak{H}$ are constructed. Then $\phi(g, e)$ can be separated from the identity by a representation $W$ of the compact group $\mathfrak{U}(n) \rtimes \mathfrak{H}$ and the desired representation $V$ of $K$ is induced from the representation $W \circ \phi$ of $\ker \Sigma$.

If $G$ is an Abelian group, then we can identify the character group $X$ of $G$ with $\hat{G}$ and with the notation as in 2 above, $F(X, T)$ is a subgroup of $X$.

**Theorem 4.** Let $V$ be a normal subgroup of a topological group $K$. Assume further that $V$ is topologically isomorphic to the additive group
of a finite-dimensional vector space over some locally compact, nondiscrete field of characteristic zero. Let $C$ be the centralizer of $V$ in $K$. Then $K$ is MAP if and only if $C$ is MAP and $K/C$ is a finite group.

We make use of Pontrjagin's classification of locally compact fields [5, Satz 22] and the fact that the field of real numbers and the $p$-adic number fields are self-dual to show that the finitely orbited characters of $V$ form a subspace of $V$, so that $F(V, T)$ is closed in $V$. Furthermore, it follows from Theorem 1 that $F(V, T)$ is dense in $V$. These facts imply that $T$ must be finite so that $C$ must have finite index in $K$.

Using a $p$-series field, a group can be constructed to show that the hypothesis above (that the field have characteristic zero) is essential.

References


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