THE OBSTRUCTION TO FIBERING A MANIFOLD OVER A CIRCLE

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1. **Introduction.** In [1], Stallings considers the following question. When does a 3-manifold fiber a circle? Browder and Levine generalized Stallings' result to differentiable and piecewise linear manifolds M of dimension greater than five under the restriction that $\pi_1(M) \cong Z$. Their theorem is purely *homotopic* in nature. That is if $h: M' \to M$ is a homotopy equivalence and $f: M \to S^1$ is a smooth fiber map then there always exists a smooth fiber map $f': M' \to S^1$ such that f' is homotopic to $f \circ h$.

This result is *false* if we drop their restriction on the fundamental group. In particular let N be the cartesian product of a 3-dimensional lens space L with fundamental group Z_{p^2} and the torus T^{n-3} where $n \geqq 5$. Let $M = N \times S^1$ and $f: M \to S^1$ denote projection onto the second factor. Then there exists a manifold M' and a homotopy equivalence $h: M' \to M$ (in fact we may take M' to be h-cobordant to M) such that a smooth fiber map $f': M' \to S^1$ homotopic to $f \circ h$ cannot exist. This example is based on recent deep results of Bass and Murthy [3] concerning the structure of the projective class group. In a joint paper with W. C. Hsiang [4] we use this example to construct an *h-cobordism* (W, M, M') *which is not homeomorphic to* $M \times [0, 1]$.

In this paper we will state necessary and sufficient conditions, in terms of a new obstruction theory, for a manifold M^n $(n \geqq 6)$ to fiber a circle. No restrictions will be placed on the fundamental group of M. We will always work in the differential category, but the corresponding theorem is also true in the piecewise-linear category.

2. **Description of obstructions.** Let M^n be a closed connected smooth manifold with $n \geqq 6$. Let $f: M \to S^1$ be a continuous map. (Recall that the homotopy class of f is an element of $H^1(M, Z)$.) We will state three properties about f which are necessary and sufficient to guarantee the existence of a smooth fiber map $\bar{f}: M \to S^1$ homotopic to f. For convenience we restrict our attention to maps f such that $f_\#: \pi_1(M) \to \pi_1(S^1)$ is onto. (This is equivalent to considering only indi-

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visible elements in $H^1(M, Z)$.) This corresponds geometrically to considering fibrations with connected fiber. Let $G=\ker f_\#$ and X denote the covering space of M corresponding to G. If $\bar{f}$ exists then it is clear that the fiber of $\bar{f}$ is homotopically equivalent to X. But the fiber of $\bar{f}$ would be a closed smooth manifold. In particular it would be a finite C.W. complex. Hence we obtain

CONDITION 1. X is dominated by a finite C.W. complex.

Let (N^{n-1}, v) be a framed submanifold of M which represents f under the Pontrjagin-Thom construction. Let M_N denote the manifold obtained by "cutting" M along N. Then ∂M_N consists of two copies of N which we label N' and N''. (See Figure 1.)

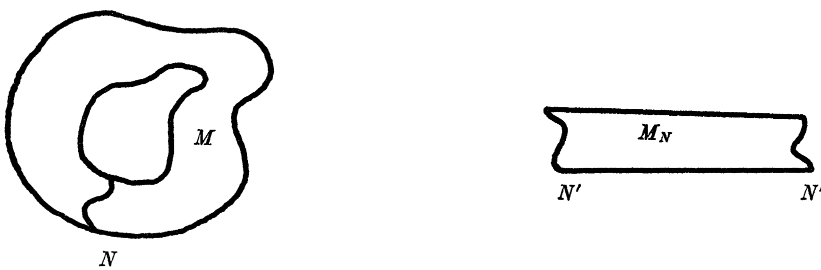

FIGURE 1

M_N is a cobordism from N' to N''. The pair (N, v) is called a splitting of M. Let $1<s<n-2$, s an integer. When Condition 1 holds it is always possible to find a splitting (N, v) such that (M_N, N') has a handlebody decomposition with handles of only two dimension s and $s+1$. The proof of this uses essentially the same arguments as in [2]. Note that the existence of a smooth fiber map is equivalent to the existence of a splitting (N, v) such that M_N is diffeomorphic to $N\times[0, 1]$. Conditions 2 and 3 will guarantee the existence of such a splitting. Condition 2 will hold if and only if there exists a splitting (N, v) such that (M_N, N', N'') is an h-cobordism. Condition 3 will hold if and only if *some* such h-cobordism is a product.

We proceed to formulate Condition 2. From the exact sequence $0\rightarrow G\rightarrow\pi_1(M)\rightarrow Z\rightarrow 0$ we see that $\pi_1(M)$ is a semidirect product of G and Z with respect to an automorphism α of G. (α is only well defined up to an inner automorphism but this is all right for our purposes.) If Condition 1 is satisfied then we can define an element $c(f)$ in an abelian group $C(Z(G), \alpha)$. $c(f)$ has the following property: $c(f)=0$ if and only if there exists a splitting (N, v) such that (M_N, N', N'') is an

h-corbodism. The proof of this fact is quite long and relies heavily on handle body theory.

CONDITION 2. $c(f)=0$.

If Condition 2 is satisfied then $\tau(M_N, N')\in \mathrm{Wh}(G)$ is defined. But it may be possible to have a second splitting (N_1, v_1) such that (M_{N_1}, N_1', N_1'') is an h-cobordism and $\tau(M_{N_1}, N_1')\neq\tau(M_N, N')$. Let α_* denote the automorphism of $\mathrm{Wh}(G)$ induced by α (see [5]). Let $\tau(f)$ be the image of $\tau(M_N,N')$ in the group $\mathrm{Wh}(G)/\{x-\alpha_*(x)\,|\,x\in \mathrm{Wh}(G)\}$ under the quotient homomorphism. We can show that $\tau(f)$ is well defined. (i.e. $\tau(f)$ is independent of the splitting (N, v)). Also $\tau(f)=0$ if and only if there exists a splitting (N, v) such that $\tau(M_N, N')=0$. The proof of this fact makes use of Stallings' realizability theorem for h-cobordisms (see [6]). But the s-cobordism theorem of Barden-Mazur-Smale states that M_N is diffeomorphic to $N\times[0, 1]$ if and only if $\tau(M_N, N')=0$. Therefore

CONDITION 3. $\tau(f)=0$.

Summarizing we have the following theorem.

THEOREM. *There exists a smooth fiber map $\bar{f}$: $M\rightarrow S^1$ homotopic to f if and only if*

1. *X is dominated by a finite* C.W. *complex,*
2. $c(f)=0$,
3. $\tau(f)=0$.

NOTE. There exists a version of this theorem for manifolds with boundary where the boundary already fibers a circle.

3. **Properties of** $C(R, \alpha)$. *If* R is a ring with identity and α is an automorphism of R then by a Grothendieck construction we can define an abelian group $C(R, \alpha)$. $\tilde{K}_0(R)$ is a direct summand of $C(R, \alpha)$. Denote by $\tilde{C}(R, \alpha)$ the complementary summand. Write $c(f)=\sigma(f)+\tilde{c}(f)$ where $\sigma(f)\in\tilde{K}_0(R)$ and $\tilde{c}(f)\in\tilde{C}(R,\alpha)$. Then $\sigma(f)$ is the Novikov-Siebenmann-Wall obstruction to X splitting differentiably as a cartesian product $N\times \boldsymbol{R}$.

R is called regular if it is Noetherian and every finitely generated R module has a resolution of finite length by projective R modules. If R is regular then $\tilde{C}(R, \alpha)=0$. But this is not the general situation since Bass and Murthy have shown that $\tilde{C}(Z(G), \mathrm{id})\neq 0$ for certain finitely generated abelian groups G. A particular example is $G=Z\oplus Z\oplus Z_4$.

As an example of the fibering theorem consider the case where $G=Z^n$. Then $Z(G)$ is regular and hence $\tilde{C}(Z(G), \alpha)=0$. Also it is known that $\tilde{K}_0(Z(G))=0$ and $\mathrm{Wh}(G)=0$ (see [**5**]). Therefore Conditions 2 and 3 become vacuous. Also observe that Condition 1 is only a homotopy theoretic condition. In particular if M and M' are homotopically equivalent manifolds such that $\pi_1(M)$ is free abelian then M fibers a circle if and only if M' fibers a circle.

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