# AN EXAMPLE IN THE FIXED POINT THEORY OF POLYHEDRA[1]

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Communicated by Felix Browder, May 29, 1967

**1. Introduction.** A finite polyhedron is constructed which enables us to answer the following questions in the negative.

(1) Is the fixed point property a homotopy type invariant in the category of finite polyhedra?

(2) Is the fixed point property a product invariant in the category of finite polyhedra, i.e. if $K_1$ and $K_2$ have the fixed point property, does $K_1 \times K_2$ have the fixed point property?

The author is indebted to Professor Edward Fadell for bringing these questions to his attention, and pointing out that if one found a polyhedron with the fixed point property and yet admitted a map with *even* Lefschetz number, then these questions could be answered.

**2. The example.** Let

$$X = P_2(C) \cup S_1 \times S_2 \cup P_4(C)$$

where $P_2(C)$ and $P_4(C)$ are complex projective spaces, $S_1$ and $S_2$ are 2-spheres, and the following identifications are made. Letting $(b_1, b_2) \in S_1 \times S_2$ be a base point, $P_1(C) \subset P_2(C)$ is identified with $S_1 \times b_2$ and $P_1(C) \subset P_4(C)$ is identified with $b_1 \times S_2$.

The cohomology ring structure of $X$ over the rational field $Q$ is given by:

$$\begin{array}{ll}
H^0(X; Q) = Q, & \text{with generator } 1,\\
H^2(X; Q) = Q \oplus Q, & \text{with generators } \alpha, \beta,\\
H^4(X; Q) = Q \oplus Q \oplus Q, & \text{with generators } \alpha^2, \alpha\beta, \beta^2,\\
H^6(X; Q) = Q, & \text{with generator } \beta^3,\\
H^8(X; Q) = Q, & \text{with generator } \beta^4.
\end{array}$$

All odd cohomology is zero, and furthermore, $\alpha^3 = \alpha^4 = \alpha\beta^2 = \alpha^2\beta = 0$. Note that $\chi(X) = 8$. ($\chi$ denotes Euler characteristic.)

THEOREM 1. *$X$ has the fixed point property.*

[1] This research was supported by the National Science Foundation under Grant GP-6682 and constitutes part of the author's dissertation written under the direction of Professor Edward Fadell.

PROOF. Let $f\colon X \to X$ denote any map and suppose $f^*(\alpha)=a\alpha+b\beta$, $f^*(\beta)=c\alpha+d\beta$. Let $g$ denote the composite

$$P_4(C) \xrightarrow{i} X \xrightarrow{f} X \xrightarrow{r} P_2(C)$$

where $r$ is the retraction which sends $S_1 \times S_2$ onto $S_1 \times b_2$ and $P_4(C)$ onto $(b_1, b_2)$. If $\alpha_1 \in H^2(P_2(C))$ and $\beta_1 \in H^2(P_4(C))$ are generators, then $g^*(\alpha_1)=b\beta_1$ and $g^*(\alpha_1^3)=b^3\beta_1^3=0$. Thus $b=0$ and hence the Lefschetz number of $f$ is given by

$$\begin{aligned} L(f) &= 1 + a + d + a^2 + ad + d^2 + d^3 + d^4 \\ &= (a + \tfrac{1}{2} + d/2)^2 + \tfrac{1}{4}(4d^4 + 4d^3 + 3d^2 + 2d + 3). \end{aligned}$$

If we let

$$p(d) = 4d^4 + 4d^3 + 3d^2 + 2d + 3, \qquad p'(d) = 2\,(2d + 1)(4d^2 + d + 1)$$

and we see that $p(d) \geqq p(-1/2) = 5/2$ and hence $L(f) > 0$.

3. **Consequences.** We first recall a theorem of Wecken [3, Theorem 2] which may be stated, in part, as follows:

THEOREM W. *Let $K$ be a finite polyhedron with the property that no finite collection of points separates $K$. Then $K$ admits a fixed point free map (homotopic to identity) if $\chi(K)=0$.*

Let $Y=\Sigma P_8(C)$ be the suspension of complex projective 8-space. A simple argument using Steenrod squares shows that $Y$ has the fixed point property. Since $\chi(Y)=-7$, $\chi(X \vee Y)=0$.

THEOREM 2. *$X \vee Y$ is a finite polyhedron of Euler characteristic 0 with the fixed point property.*

THEOREM 3. *$X$ and $Y$ are two finite polyhedra with the fixed point property such that their union along an edge fails to have the fixed point property.*

Just apply Theorem W to $X \cup_I Y$, the union of $X$ and $Y$ joined along an edge.

Since $X \vee Y$ and $X \cup_I Y$ are of the same homotopy type, we have the following:

COROLLARY. *The fixed point property is not a homotopy type invariant in the category of finite polyhedra.*

Now, let $Z = X \vee Y$. Applying Theorem W to $Z \times I$ and $Z \times Z$ we obtain

THEOREM 4. *$Z$ is a polyhedron with the fixed point property such that $Z\times I$ and $Z\times Z$ fail to have the fixed point property. Thus the fixed point property is not a product invariant in the category of finite polyhedra.*

Note that $\chi(\Sigma^2 Z)=0$ and hence, using Theorem W again, the double suspension $\Sigma^2 Z$ admits a fixed point free map. $\Sigma Z$ either admits a fixed point free map or has the fixed point property. In either case, we obtain

THEOREM 5. *There is a finite polyhedron $K$ with the fixed point property such that $\Sigma K$ fails to have the fixed point property.*

REMARKS. Note that all the above examples are simply connected. We might also mention that, using [1] and [2], the fixed point property *is* a homotopy type invariant in the category of polyhedra of $\dim > 2$ having the homotopy type of simply connected, closed topological manifolds.

## REFERENCES

1. E. Fadell, *On a coincidence theorem of F.B. Fuller*, Pacific J. Math. (3) **15** (1965), 825–834.

2. Shi Gen-Hua, *On least number of fixed points and Nielsen numbers*, Chinese Math. **8** (1966), 234–243.

3. Wecken, *Fixpunktklassen*. III, Math. Ann. **118–119** (1941–1943), 544–577.

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