AN EXAMPLE IN THE FIXED POINT THEORY OF POLYHEDRA

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1. Introduction. A finite polyhedron is constructed which enables us to answer the following questions in the negative.

(1) Is the fixed point property a homotopy type invariant in the category of finite polyhedra?

(2) Is the fixed point property a product invariant in the category of finite polyhedra, i.e. if $K_1$ and $K_2$ have the fixed point property, does $K_1 \times K_2$ have the fixed point property?

The author is indebted to Professor Edward Fadell for bringing these questions to his attention, and pointing out that if one found a polyhedron with the fixed point property and yet admitted a map with even Lefschetz number, then these questions could be answered.

2. The example. Let

$$X = P_2(C) \cup S_1 \times S_2 \cup P_4(C)$$

where $P_2(C)$ and $P_4(C)$ are complex projective spaces, $S_1$ and $S_2$ are 2-spheres, and the following identifications are made. Letting $(b_1, b_2) \in S_1 \times S_2$ be a base point, $P_1(C) \subset P_2(C)$ is identified with $S_1 \times b_2$ and $P_1(C) \subset P_4(C)$ is identified with $b_1 \times S_2$.

The cohomology ring structure of $X$ over the rational field $\mathbb{Q}$ is given by:

- $H^0(X; \mathbb{Q}) = \mathbb{Q}$, with generator 1,
- $H^2(X; \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$, with generators $\alpha, \beta$,
- $H^4(X; \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$, with generators $\alpha^2, \alpha \beta, \beta^2$,
- $H^6(X; \mathbb{Q}) = \mathbb{Q}$, with generator $\beta^3$,
- $H^8(X; \mathbb{Q}) = \mathbb{Q}$, with generator $\beta^4$.

All odd cohomology is zero, and furthermore, $\alpha^2 = \alpha^4 = \alpha \beta^2 = \alpha^2 \beta = 0$.

Note that $\chi(X) = 8$. ($\chi$ denotes Euler characteristic.)

Theorem 1. $X$ has the fixed point property.

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PROOF. Let \( f: X \rightarrow X \) denote any map and suppose \( f^*(\alpha) = a\alpha + b\beta, \)
\( f^*(\beta) = c\alpha + d\beta. \) Let \( g \) denote the composite
\[ P_4(C) \xrightarrow{i} X \xrightarrow{f} X \xrightarrow{r} P_4(C) \]
where \( r \) is the retraction which sends \( S_1 \times S_2 \) onto \( S_1 \times b_2 \) and \( P_4(C) \)
onto \( (b_1, b_2). \) If \( \alpha_1, \beta_1 \in H^2(P_4(C)) \) and \( \beta_1 \in H^3(P_4(C)) \) are generators, then
\( g^*(\alpha_1) = b\beta_1 \) and \( g^*(\beta_2) = b^*\beta_2 = 0. \) Thus \( b = 0 \) and hence the Lefschetz
number of \( f \) is given by
\[
L(f) = 1 + a + d + a^2 + ad + d^2 + d^3 + d^4
\]
\[ = (a + \frac{1}{2} + d/2)^2 + \frac{1}{2}(4d^4 + 4d^3 + 3d^2 + 2d + 3). \]
If we let
\[
p(d) = 4d^4 + 4d^3 + 3d^2 + 2d + 3, \quad p'(d) = 2(2d + 1)(4d^2 + d + 1)
\]
and we see that \( p(d) \geq p(-1/2) = 5/2 \) and hence \( L(f) > 0. \)

3. Consequences. We first recall a theorem of Wecken [3, The­
orem 2] which may be stated, in part, as follows:

THEOREM W. Let \( K \) be a finite polyhedron with the property that no
finite collection of points separates \( K. \) Then \( K \) admits a fixed point free
map (homotopic to identity) if \( \chi(K) = 0. \)

Let \( Y = \Sigma P_4(C) \) be the suspension of complex projective 8-space.
A simple argument using Steenrod squares shows that \( Y \) has the
fixed point property. Since \( \chi(Y) = -7, \chi(X \setminus Y) = 0. \)

THEOREM 2. \( X \setminus Y \) is a finite polyhedron of Euler characteristic 0
with the fixed point property.

THEOREM 3. \( X \) and \( Y \) are two finite polyhedra with the fixed point
property such that their union along an edge fails to have the fixed point
property.

Just apply Theorem W to \( X \cup_f Y, \) the union of \( X \) and \( Y \) joined
along an edge.

Since \( X \setminus Y \) and \( X \cup_f Y \) are of the same homotopy type, we have
the following:

COROLLARY. The fixed point property is not a homotopy type invariant
in the category of finite polyhedra.

Now, let \( Z = X \setminus Y. \) Applying Theorem W to \( Z \times I \) and \( Z \times Z \)
we obtain
THEOREM 4. $Z$ is a polyhedron with the fixed point property such that $Z \times I$ and $Z \times Z$ fail to have the fixed point property. Thus the fixed point property is not a product invariant in the category of finite polyhedra.

Note that $\chi(\Sigma^2 Z) = 0$ and hence, using Theorem W again, the double suspension $\Sigma^2 Z$ admits a fixed point free map. $\Sigma Z$ either admits a fixed point free map or has the fixed point property. In either case, we obtain

THEOREM 5. There is a finite polyhedron $K$ with the fixed point property such that $\Sigma K$ fails to have the fixed point property.

REMARKS. Note that all the above examples are simply connected.

We might also mention that, using [1] and [2], the fixed point property is a homotopy type invariant in the category of polyhedra of dim$\geq 2$ having the homotopy type of simply connected, closed topological manifolds.

REFERENCES


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