GENERATORS FOR SOME RINGS OF
ANALYTIC FUNCTIONS

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Let $\Omega$ be an open set in $\mathbb{C}^n$ and let $\rho$ be a nonnegative function defined in $\Omega$. We shall denote by $A_\rho(\Omega)$ the set of all analytic functions $f$ in $\Omega$ such that for some constants $C_1$ and $C_2$

$$|f(z)| \leq C_1 \exp(C_2 \rho(z)), \quad z \in \Omega.$$  

(1)

It is obvious that $A_\rho(\Omega)$ is a ring. We wish to determine when it is generated by a given finite set of elements $f_1, \ldots, f_N$. There is an obvious necessary condition, for if $f_1, \ldots, f_N$ are generators for $A_\rho(\Omega)$ we can in particular find $g_1, \ldots, g_N \in A_\rho(\Omega)$ so that $1 = \sum f_j g_j$. Hence we have

$$1 \leq \sum |f_j(z)| C_1 \exp(C_2 \rho(z))$$

for some constants $C_1$ and $C_2$, that is,

$$|f_1(z)| + \cdots + |f_N(z)| \geq c_1 \exp(-c_2 \rho(z)), \quad z \in \Omega,$$

(2)

for some positive constants $c_1$ and $c_2$.

This note concerns the converse statement. Carleson [1] has proved a deep result of that type, called the Corona Theorem, which states that (2) implies that $f_1, \ldots, f_N$ generate $A_\rho(\Omega)$ if $\rho = 0$ and $\Omega$ is the unit disc in $\mathbb{C}$. In a recent research announcement [5] in this Bulletin, the Corona Theorem was used to prove the analogous result when $\rho(z) = |z|$ and $\Omega = \mathbb{C}$. However, we shall see here that this statement is much more elementary than the Corona Theorem; indeed, we shall prove a general result of this kind for functions of several complex variables although no analogue of the Corona Theorem is known there.

**Theorem 1.** Let $\rho$ be a plurisubharmonic function in the open set $\Omega \subset \mathbb{C}^n$ such that

(i) all polynomials belong to $A_\rho(\Omega)$;

(ii) there exist constants $K_1, \ldots, K_4$ such that $z \in \Omega$ and $|z - \xi| \leq \exp(-K_4 \rho(z) - K_3) \Rightarrow \xi \in \Omega$ and $\rho(\xi) \leq K_2 \rho(z) + K_4$.

Then $f_1, \ldots, f_N \in A_\rho(\Omega)$ generate $A_\rho(\Omega)$ if and only if (2) is valid.

Before the proof we make a few remarks. First note that if $d(z)$
denotes the distance from \( z \in \Omega \) to \( \partial \Omega \) then (ii) implies that \( d(z) \geq \exp(-K_1\rho(z) - K_2) \), that is,

\[
\rho(z) \geq \frac{\log 1/d(z) - K_2}{K_1}.
\]

Hence \( \rho(z) \to \infty \) if \( z \) converges to a boundary point of \( \Omega \), so \( \Omega \) is pseudo-convex and therefore a domain of holomorphy (cf. [3, Theorem 4.2.8]). On the other hand, if \( \Omega \) is a domain of holomorphy it follows that \( \rho(z) = \log 1/d(z) \) is plurisubharmonic, and (ii) is valid with \( K_1 = K_3 = 1 \) and suitable \( K_2, K_4 \). Another example is obtained by taking \( \rho(z) = \sum |z_j|^p, \Omega = \mathbb{C}^n \), where \( p \) is any positive number. When \( n = 1 \) this yields the results announced in [5]. However, the Corona Theorem is not contained in Theorem 1 but will be discussed at the end of the note.

We know already that (2) is a necessary condition for \( f_1, \ldots, f_N \) to be generators. To prove the sufficiency we shall apply a standard homological argument (cf. e.g. Malgrange [6]) but first a few lemmas are required.

**Lemma 2.** If \( f \in A_p(\Omega) \) it follows that \( \partial f/\partial z \in A_p(\Omega) \).

**Proof.** From (1) and (ii) we obtain

\[
|f(\xi)| \leq C_1 \exp(C_2(K_3\rho(z) + K_4)) \quad \text{if} \quad |\xi - z| \leq \exp(-K_1\rho(z) - K_2).
\]

Hence

\[
|\partial f(z)/\partial z| \leq C_1 \exp(C_2(K_3\rho(z) + K_4) + K_1\rho(z) + K_2).
\]

Since we shall use \( \bar{\partial} \) cohomology with bounds in \( L^2 \) norms, we also note that the definition of \( A_p(\Omega) \) can be expressed in terms of such norms.

**Lemma 3.** If \( f \) is analytic in \( \Omega \), then \( f \in A_p(\Omega) \) if and only if for some \( K \)

\[
(3) \quad \int |f|^2 e^{-2K_p}\lambda < \infty,
\]

where \( d\lambda \) denotes the Lebesgue measure.

**Proof.** If (1) is valid we obtain (3) since \( (1 + |z|)^{2n+1} \leq B_1 \exp B_2\rho(z) \) in view of (i). On the other hand, it follows from (3) and (ii) that the mean value of \( |f| \) over the ball \( \{ \xi; |\xi - z| \leq \exp(-K_1\rho(z) - K_2) \} \) is bounded by \( C \exp(K(K_3\rho(z) + K_4) + 2n(K_1\rho(z) + K_2)) \). Since this is also a bound for \( |f(z)| \), the lemma is proved.
Lemma 4. Let \( g \) be a form of type \((0, r+1)\) in \( \Omega \) with locally square integrable coefficients and \( \bar{\partial}g = 0 \), and let \( \phi \) be a plurisubharmonic function in \( \Omega \) such that

\[
\int |g|^2 e^{-\phi} d\lambda < \infty.
\]

If \( r \geq 0 \) it follows that there is a form \( f \) of type \((0, r)\) with \( \bar{\partial}f = g \) and

\[
\int |f|^2 e^{-\phi}(1 + |z|^2)^{-2} d\lambda \leq \int |g|^2 e^{-\phi} d\lambda.
\]

The norms here are defined as in §4.1 of [3]. The lemma follows from Theorem 2.2.1' in [2] by the argument used in [3] to derive Theorem 4.4.2 from Theorem 4.4.1.

For nonnegative integers \( s \) and \( r \) we shall denote by \( L^s_r \) the set of all differential forms \( h \) of type \((0, r)\) with values in \( \Lambda^s \mathbb{C}^N \), such that for some \( K \)

\[
\int |h|^2 e^{-2Kp} d\lambda < \infty.
\]

In other words, for each multi-index \( I = (i_1, \ldots, i_s) \) of length \( |I| = s \) with indices between 1 and \( N \) inclusively, \( h \) has a component \( h_I \) which is a differential form of type \((0, r)\) such that \( h_I \) is skew symmetric in \( I \) and

\[
\int |h_I|^2 e^{-2Kp} d\lambda < \infty.
\]

The \( \bar{\partial} \) operator defines an unbounded map from \( L^s_r \) to \( L^{s+1}_r \); its domain consists of all \( h \in L^s_r \) such that \( \bar{\partial}h \), defined in the sense of distribution theory with \( \bar{\partial} \) acting on each component \( h_I \) is an element of \( L^{s+1}_r \). Furthermore, the interior product \( P_I \) by \((f_1, \ldots, f_N)\) maps \( L^{s+1}_r \) into \( L^s_r \); If \( h \in L^s_r \) then

\[
(P_I h)_{I} = \sum_1^N h_{Ij} f_j,
\]

We define \( P_I L^0_r = 0 \). Clearly \( P_I^2 = 0 \) and \( P_I \) commutes with \( \bar{\partial} \) since \( f_j \) are analytic, so we have a double complex.

Lemma 5. The equation \( \bar{\partial}g = h \) has a solution \( g \in L^s_r \) for every \( h \in L^s_{r+1} \) with \( \bar{\partial}h = 0 \).

Proof. In view of (i) this is an immediate consequence of Lemma 4.
**Lemma 6.** If \( g \in L^r \) and \( Pf g = 0 \), we can find \( h \in L^{r+1} \) such that \( g = Pf h \) and in addition \( \partial h \in L^s \) if \( \bar{g} = 0 \).

**Proof.** We can take for \( h \) essentially the exterior product of \( g \) by \( \bar{f}/|f|^2 \). More precisely, we set when \( |I| = s + 1 \)

\[
h_I = \sum_{i=1}^{s+1} g_i(-1)^{s+1-i} |\bar{f} + i| / |f|^2,
\]

where \( I_j \) denotes the multi-index \( I = (i_1, \ldots, i_{s+1}) \) with the index \( i_j \) removed. It follows from (2) that \( h \in L^{r+1}_r \), and since \( Pf g = 0 \) it is obvious that \( Pf h = g \). If \( \bar{g} = 0 \) we can compute \( \partial h_I \) by operating on the factor \( \bar{f}/|f|^2 \) alone, so it follows from (2) and Lemma 2 that \( \partial h \in L^{s+1}_s \).

It is now easy to prove the following theorem which in view of Lemma 3 contains Theorem 1 for \( r = s = 0 \). (Actually Theorems 1 and 7 are equivalent.)

**Theorem 7.** For every \( g \in L^r \) with \( \bar{g} = Pf g = 0 \) one can find \( h \in L^{r+1} \) so that \( \partial h = 0 \) and \( Pf h = g \).

**Proof.** The theorem is trivially valid when \( r > n \) or \( s > N \). In the proof we may therefore assume that it has already been established for larger values of \( r \) and \( s \). By Lemma 6 we can find \( h' \in L^{s+1}_s \) so that

\[
Pf h' = g, \quad \partial h' \in L^{s+1}_s.
\]

Since \( \bar{\partial} h' = 0 \) and \( Pf \bar{\partial} h' = \bar{Pf} h' = \bar{g} = 0 \), it follows from the inductive hypothesis that one can find \( h'' \in L^{s+2} \) such that

\[
Pf h'' = \bar{\partial} h', \quad \bar{\partial} h'' = 0.
\]

By Lemma 5 we can find \( h''' \in L^{s+2}_s \) so that \( \bar{\partial} h''' = h'' \). If \( h = h' - Pf h''' \) we conclude that \( \bar{\partial} h = \bar{\partial} h' - Pf \bar{\partial} h''' = \bar{\partial} h' - Pf h'' = 0 \), and that \( Pf h = Pf h' = g \). The proof is complete.

We shall end this note by showing how the proofs of Carleson [1] can be adapted to the conventional pattern used in the proof of Theorem 1. This does not remove the main difficulties but it does eliminate a tricky argument due to D. J. Newman, which was used in [1] in the case of more than 2 generators. In the proof of Theorem 1 the main points were the existence theorems for the operators \( \bar{\partial} \) and \( Pf \) given in Lemmas 5 and 6. The proof of the Corona Theorem requires a more precise version of both.

From now on \( \Omega \) will denote the unit disc in \( \mathbb{C} \). (All the arguments are valid for any bounded open set in \( \mathbb{C} \) with a \( C^2 \) boundary.) If \( \mu \) is a
bounded measure in $\Omega$ and $\phi$ is an integrable function on $\partial \Omega$, we shall say that a distribution in $\Omega$ satisfying the Cauchy-Riemann equation

\begin{equation}
\frac{\partial u}{\partial \bar{z}} = \mu \text{ in } \Omega
\end{equation}

has boundary values $\phi$ on $\partial \Omega$ provided that there exists a distribution $U$ with support in $\bar{\Omega}$ such that $U = u$ in $\Omega$ and

\begin{equation}
\frac{\partial U}{\partial \bar{z}} = \mu - \phi dz/2i.
\end{equation}

Here $\phi dz$ is of course a measure on $\partial \Omega$, and $\mu$ is extended so that there is no mass in the complement of $\Omega$. If $u = 0$ it follows from (6) that $U = 0$, for $\partial U/\partial \bar{z}$ would otherwise be a distribution with support on $\partial \Omega$ with positive transversal order. Hence $u$ determines both $\phi$, $\mu$ and $U$ uniquely, so it is legitimate for us to say that $\phi$ is the boundary value of $u$.

If $u$ belongs to the Hardy class $H^p$ for some $p \geq 1$, then $\phi$ coincides a.e. with the boundary values in the usual sense, and $\mu = 0$. Conversely, if $u$ is analytic and has boundary values belonging to $L^p(\partial \Omega)$ in the sense of (6), it follows that $u \in H^p$ ($p \geq 1$). If $f \in H^\infty$ and $u$ is a solution of (5) with boundary values $\phi$, then $fu$ satisfies (5) with $\mu$ replaced by $f\mu$ and has boundary values $f\phi$. This is obvious when $f$ is analytic in a neighborhood of $\bar{\Omega}$ and follows in general if we first consider $f(rz)$ with $r < 1$ and then let $r \to 1$, noting that the solution $U \in \mathcal{E}'(\Omega)$ of the equation $\partial U/\partial \bar{z} = F$ is a continuous function of $F \in \mathcal{E}'(\Omega)$ when it exists.

The existence of a solution of (6) with support in $\bar{\Omega}$ means precisely that the right hand side is orthogonal to all (entire) analytic functions. Thus (5) has a solution with boundary values $\phi$ if and only if for entire analytic $f$

\[
\int f d\mu = (2i)^{-1} \int \phi(z)f(z)dz.
\]

In view of the Hahn-Banach Theorem it follows that there exists a solution with boundary values of absolute value $\leq C$ if and only if for entire analytic $f$

\[
\left| \int f d\mu \right| \leq C \int |f(z)| |dz| /2.
\]

A sufficient condition for this is given by the following result of [1]. (See also [4] where an extension to several variables is given.)

**Lemma 8.** There is a constant $C$ such that
for every measure $\mu$ in $\Omega$ such that
\[ (8) \quad |\mu| \{ \xi; |\xi - z| < r \} \leq Mr, \quad z \in \partial \Omega, \quad r > 0. \]

We now modify the definition of $L^s_r$ as follows:

$h \in L^s_0$ if $\partial h/\partial \bar{z}$ is a bounded measure in $\Omega$ and $h_I$ has boundary values in $L^\infty(\partial \Omega)$, $|I| = s$; $h \in L^s_1$ if $h_I = \mu_id\bar{z}$ where $\mu_I$ is a measure in $\Omega$ satisfying (8), $|I| = s$. Of course we take $L^s_r = 0$ when $r > 1$. From Lemma 8 and the discussion preceding it we conclude that Lemma 5 remains valid and that $\{ h; h \in L^s_0, \partial h = 0 \} = H^\infty$.

Let $f_j \in H^\infty, j = 1, \ldots, N$, and assume that for some $c > 0$
\[ (2)' \quad |f_1| + \cdots + |f_N| \geq c. \]

If we define $P_f$ by means of these functions, the proof of Lemma 6 remains valid when $s = 1$ but breaks down when $s = 0$ since $\partial f_j/\partial z$ need not be a bounded function. We must therefore use another construction, based on the following

**Lemma 9.** For sufficiently small $\varepsilon > 0$ one can find a partition of unity $\phi_j$ subordinate to the covering of $\Omega$ by the open sets $\Omega_j = \{ \xi; |f_j(\xi)| > \varepsilon \}$ such that $\partial \phi_j/\partial \bar{z}$, defined in the sense of distribution theory, is a measure which satisfies (8) for all $j$ and some $M$.

Admitting Lemma 9 for a moment we shall see that it implies the Corona Theorem. With our new definition of $L^s_r$ we have already seen that Lemma 5 remains valid as well as Lemma 6 for $r \neq 0$. To prove Lemma 6 for $r = 0$ we need only replace $f_j/|f_j|^2$ in the previous proof by $\phi_j/f_j$ where $\phi_j$ is the partition of unity in Lemma 9. In fact, $\partial(\phi_j/f_j)/\partial \bar{z} = f_j^{-1}\partial \phi_j/\partial \bar{z}$ satisfies (8) since $|f_j| \geq \varepsilon$ in supp $\phi_j$. Hence the proof of Theorem 7 can be applied without change. For $r = s = 0$ we obtain the only interesting conclusion:

**Theorem 10. (The Corona Theorem).** If $f_1, \ldots, f_N \in H^\infty$ and (2)' is valid, it follows that $f_1, \ldots, f_N$ are generators for $H^\infty$.

It remains to discuss the proof of Lemma 9. Since the set of bounded functions $\psi$ with $\partial \psi/\partial \bar{z}$ satisfying (8) is a ring, the standard technique for constructing partitions of unity can be applied to derive Lemma 9 from

**Lemma 11.** There exists a constant $k$ such that if $0 < \varepsilon < \frac{1}{2}$ and $f \in H^\infty$, $\sup |f| \leq 1$, one can find $\psi$ with $0 \leq \psi \leq 1$ so that $\partial \psi/\partial \bar{z}$ satisfies (8) and
\[ \psi(z) = 0 \text{ when } |f(z)| < \varepsilon, \quad \psi(z) = 1 \text{ when } |f(z)| > \varepsilon. \]

This lemma was proved in a different formulation in [1] when \( f \) is a Blaschke product. In fact, the main point in [1] is a construction of certain curves \( \Gamma \) surrounding the zeros of a Blaschke product and satisfying conditions which mean precisely that the characteristic function \( \psi \) of the exterior of \( \Gamma \) has the properties stated in Lemma 11. Since the proof given in [1] is applicable to arbitrary \( f \in H^\infty \) and we have no significant simplification to contribute, we shall not carry out the proof here.

**References**

2. L. Hörmander, *\( L^p \) estimates and existence theorems for the \( \overline{\partial} \) operator*, Acta Math. 113 (1965), 89–152.