A COMBINATORIAL COINCIDENCE PROBLEM

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Communicated by N. Levinson, July 7, 1967

Let $A \subset E^n$ ($n \geq 1$), let $B(o) \subset E^n$ be convex with center of symmetry at $o$, let $n$ and $p$ be integers ($1 \leq p \leq n$, $n \geq 2$), and let $f(u)$ be an integrable function defined on $A$. Let $A^n$ be the Cartesian product of $A$ with itself $n$ times and define $Y \subset A^n$ by

$$Y = \left\{ x = (x_1, \ldots, x_n) : \bigcap_{k=1}^p B(x_{i_k}) \neq \emptyset \right\}$$

for some $i_1, \ldots, i_p$, $1 \leq i_1 < \ldots < i_p \leq n$.

The problem of evaluating $J = \int_{A^n} \prod_{i=1}^n f(x_i) dx_1 \cdots dx_n$ generalizes a number of questions in probability, queuing theory, scattering, statistical mechanics etc., [1], [2]. Put

$$M = \binom{n}{p}, \quad S_{i_1 \ldots i_p} = \left\{ (x_1, \ldots, x_n) : \bigcap_{i=1}^p B(x_{i_k}) \neq \emptyset \right\}, F(x)$$

$$= \prod_{i=1}^n f(x_i), \quad dV = dx_1 \cdots dx_n$$

and let the $M$ sets $S_{i_1 \ldots i_p}$ be enumerated as $\{S_k\}$, $k = 1, \ldots, M$. Then by the inclusion-exclusion principle [2]

$$J = \sum_{r=1}^{n} (-1)^{r+1} \left[ \sum_{1 \leq k_1 < \ldots < k_r \leq M} \int_{S_{k_1} \cap \ldots \cap S_{k_r}} F(x) dV \right]$$

$$= \sum_{r=1}^{n} (-1)^{r+1} U_r,$$

say. To help us keep track of different $r$-tuples of $p$-tuples, we introduce a generalization of graphs. Let $X$ be a regular simplex in $E^{n-1}$, with the vertices $w_1, \ldots, w_n$, a $(d$-dimensional) hypergraph $G$ on $X$ is just a collection of some of the $C_d^n$ $d$-dimensional faces of $X$; the number of vertices of $X$ lying in $G$ will be denoted by $v(G)$. $G$ is called a $(B, r)$-hypergraph on $X$ if it consists of $r$ such $d$-faces and if there are some $v = v(G)$ translates $B_1, \ldots, B_v$ of $B$ such that any $d+1$ of them, say $B_1, \ldots, B_{d+1}$, intersect if the corresponding vertices $w_1, \ldots, w_{d+1}$ lie in a $d$-face of $X$ included in $G$.

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$G$ is called connected if no hyperplane in $E^{n-1}$ strictly separates some of its $d$-faces from the rest of them. Let $t=t(r, d)$ be the number of types of (topologically) distinct $(B, r)$-hypergraphs on $X$, let $G_j$ be any one of the $j$th type, and let $M_j(n)$ be the number of distinct $(B, r)$-hypergraphs on $X$ of the $j$th type. Let $J_0 = \int_{R} f(u) du$, if $d = p - 1$ observe that each $d$-face of a $(B, r)$-hypergraph corresponds to exactly one set $S_k$; let

$$J(G) = \int_{S_k} \cdots \int_{S_k} F(x) dV$$

where $S_k, \ldots, S_k$ are the $S$-sets corresponding to the $d$-faces of $G$. Now we get a formula for the summand $U_r$ of (1):

$$(2) \quad U_r = \sum_{j=1}^{t(r, p-1)} M_{rj}^{p-1}(n) J_0 \prod_{c(G_j)} J(C(G_j))$$

where the product is taken over the connected components $C(G_j)$ of $G_j$. This generalizes some of the so-called cluster expansions of statistical mechanics [3].

In most applications it is found that $A$ and $B$ are simple regular sets (cubes, balls), $B$ is small while $A$ is large, and $f$ is well behaved. (1) and (2) allow us then, in principle at least, to expand $J$ in the powers of a parameter measuring the ratio of sizes of $B$ to $A$, and to estimate the error of truncation. The integrals $J(C(G_j))$ can rarely be found analytically but the Monte-Carlo method lends itself very well to their numerical evaluation.

The following expansions and identities for iterated binomial coefficients were found in the process of evaluating the numbers $M_{rj}^{p-1}(n)$ in (2). Let $q = q(r, d)$ be the smallest integer $\geq$ the largest positive root of $r = \sum_{d+u}^{x}$ then

$$(3) \quad \binom{n}{d} = \sum_{k=q}^{rd} A_{kr}(d) \binom{n}{k}$$

where

$$(4) \quad A_{kr}(d) = \sum_{j=0}^{k-r} (-1)^j \binom{k}{j} \binom{k-j}{d}.$$

Equating the coefficients of like powers of $n$ in (3) one gets
Introduction.

This paper is concerned with nonexpanding maps from the unit ball of a real Hilbert space into itself. Browder [1] has established that such maps always possess at least one fixed point. We shall develop a method, which resembles the simple iterative method, for approximating fixed points of such maps. In fact, we shall generate a sequence, \( \{x_n\} \), by the recursive formula

\[
x_{n+1} = k_n + f(x_n),
\]

where \( f \) is the map in question and \( \{k_n\} \) is a sequence of real numbers. Our main result is Theorem 3 which states sufficient conditions on \( k_n \) to insure the strong convergence of \( x_n \) to a fixed point of \( f \).

Definitions and preliminary observations.

Let \( H \) be a Hilbert space with inner product denoted by \( (\cdot,\cdot) \) and norm by \( \|\cdot\| \). Let \( B \) be the unit ball, \( B = \{x \in H : \|x\| \leq 1\} \). A map \( f: B \to B \) is nonexpanding if

\[
\|f(x) - f(y)\| \leq \|x - y\|
\]

for all \( x, y \in B \).

Assume that \( f: B \to B \) is nonexpanding. It is not difficult to establish that the set \( F \) of fixed points must be convex. Using the con-