A LIE PRODUCT FOR THE COHOMOLOGY OF
SUBALGEBRAS WITH COEFFICIENTS IN
THE QUOTIENT

BY ALBERT NIJENHUIS

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1. Outline. We consider an algebra (i.e. an associative algebra or a Lie algebra) $A$ and a subalgebra $B$. Then $B$, $A$ and also $A/B$ are (two-sided) $B$-modules in the obvious fashion. The exact sequence of coefficient modules

$$0 \rightarrow B \rightarrow A \xrightarrow{i} A/B \rightarrow 0$$

induces on the (graded) Hochschild [resp. Eilenberg-Mac Lane] cohomology modules the exact triangle of homomorphisms

$$
\begin{array}{ccc}
H^*(B, B) & \xrightarrow{i^*} & H^*(B, A) \\
\downarrow{\delta^*} & & \downarrow{\pi^*} \\
H^*(B, A/B) & & 
\end{array}
$$

The product operation in $B$, and similarly in $A$, induces a graded Lie algebra (GLA) structure (here called the cup structure) on $H^*(B, B)$ and $H^*(B, A)$ (cf., e.g., Gerstenhaber [2], Nijenhuis and Richardson [6]), and $i^*$ is known to be a homomorphism of these structures. The cup structure on $H^*(B, B)$ is abelian; cf. [2]. It is also known that $H^*(B, B)$ has another GLA structure (here called the comp structure) with respect to the reduced grading (elements of $H^*_n(B, B)$ have reduced degree $n-1$; cf. [2], [7]). The following theorem supplements this information.

**Theorem.** Let $A$ be an algebra, $B$ a subalgebra and let $A/B$ have its natural structure as a $B$-module. Then $H^*(B, A/B)$ has a GLA structure (cup structure). The maps $i^*$ and $\pi^*$ in the exact triangle (1) are homomorphisms of cup structures. The image of $i^*$ belongs to the center of $H^*(B, A)$. The map $\delta^*$ is a homomorphism between the cup structure of $H^*(B, A/B)$ and the comp structure of $H^*(B, B)$.

The theorem has immediate relevance for the theory of deformations. $H^1(B, A)$ is the set of infinitesimal nontrivial deformations of

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the homomorphism \( \iota \); the cup product provides the obstructions to finite deformations (cf. Nijenhuis and Richardson [6]). \( H^1(B, A/B) \) is the set of nontrivial infinitesimal deformations of \( B \) as a subalgebra of \( A \) (cf. Richardson [8]). In a forthcoming paper we shall show that the cup product provides the obstructions to finite deformations. \( H^2(B, B) \) is the set of nontrivial deformations of the structure of \( B \) (subject only to the condition that the structure remain of the same type—associative or Lie), and the comp product gives the obstructions to finite deformations (cf. Gerstenhaber [3] and Nijenhuis and Richardson [7].) The homomorphisms \( i^*, \pi^* \) and \( \delta^* \) provide the natural relationships between the infinitesimal deformations of the various kinds and the obstructions.

The origin of the formula (11) which defines the cup product in \( H^*(B, A/B) \) can be found in differential geometry, where it exists as an operation yielding a vector form (differential form with values which are tangent vectors) as the product of two vector forms through a process of differentiation without the intervention of any additional structure (e.g. a connection; cf. Nijenhuis [4] and Frölicher and Nijenhuis [1]). It has been extensively applied to deformations of complex structures. The present result may also have implications for the cohomology of foliations, as a foliation is a subalgebra of the Lie algebra of vector fields.

2. Basic formulas. Let \( B \) denote a vector space over a field \( k \). If \( f \) and \( g \) are cochains, i.e., elements of \( C^*(B, B) = \text{Hom}_k(\otimes B, B) \), of degrees \( n \) resp. \( m \), the composition product \( f \circ g \), of degree \( n+m-1 \), is defined by

\[
(f \circ g)(x_1, \ldots, x_{n+m-1}) = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f(x_1, \ldots, x_{i-1}, g(x_{i+1}, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{n+m-1}).
\]

Although generally not associative (cf. [2]),

\[
(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{(m-1)(n-1)}\{(f \circ h) \circ g - f \circ (h \circ g)\}
\]

for \( h \in C^p(B, B) \), this product has a commutator

\[
[f, g] = g \circ f - (-1)^{(m-1)(n-1)} f \circ g
\]

which defines a GLA structure (comp structure) on \( C^*(B, B) \), with respect to the reduced grading. For \( \mu \in C^0(B, B) \) the condition \( \mu \circ \mu = 0 \) (or \( [\mu, \mu] = 0 \) if \( \text{char } k \neq 2 \)) is equivalent to \( \mu \) defining an
associative algebra structure on \( B \). The Hochschild coboundary operator on \( C^\bullet(B, B) \) is given by \( \delta f = -[\mu, f]^\circ \).

Let \( A \) be an associative algebra with product map \( \mu \), and let \( B \) be a subalgebra. Then every \( f \in C^\bullet(B, A) \) is the restriction to \( B \) of some (not unique) \( \tilde{f} \in C^\bullet(A, A) \). The restriction of \(-[\mu, \tilde{f}]\) to \( B \) depends on \( f \) but not on the choice of \( \tilde{f} \), and is \( \delta f \in C^\bullet(B, A) \). For every \( f \in C^\bullet(B, A/B) \) there is a (nonunique) \( \tilde{f} \in C^\bullet(B, A) \) such that \( \pi \circ \tilde{f} = f \). Then \( \pi \circ \delta \tilde{f} \) depends on \( f \) but not the choice of \( \tilde{f} \), and is just \( \delta f \in C^\bullet(B, A/B) \).

If \( f, g \in C^\bullet(B, A) \) have degrees \( n \) resp. \( m \), then \( f \cup g \), of degree \( n+m \), is defined by (cf. [2])

\[
(f \cup g)(x_1, \cdots, x_{n+m}) = \mu(f(x_1, \cdots, x_n), g(x_{n+1}, \cdots, x_{n+m})).
\]

As this product is associative, commutators yield a GLA structure, defined thus:

\[
[f, g] = f \cup g - (-1)^{mn} g \cup f.
\]

The operator \( \delta \) acts as a derivation of degree 1 with respect to the cup structure on \( C^\bullet(B, A) \), hence induces the cup structure on \( H^\bullet(B, A) \). Also, \( h \) acts as a derivation of degree \( p - 1 \). If \( f \) has values in \( B \), then \( [f, g] \) is expressible in terms of \( \bar{\sigma} \) (cf. [2])

\[
[f, g] = (-1)^{m-1}\{(\mu \bar{\sigma} g) \bar{\sigma} f - \mu \bar{\sigma} (g \circ f)\}
= \delta g \bar{\sigma} f + (-1)^n \delta \bar{\sigma} (g \circ f) - (-1)^n g \bar{\sigma} \delta f.
\]

This provides a formula for \( \delta (g \bar{\sigma} f) \), and also shows that the cup structure on \( H^\bullet(B, B) \) is abelian. In fact, it shows the following:

**Lemma 2.1.** The image \( i^\bullet(H^\bullet(B, B)) \) belongs to the center of \( H^\bullet(B, A) \) with respect to the cup structure.

A second complex, \( C^\bullet(B, B) \) Hom\(_k\) (\( AB, B \)) has a composition product, usually called the hook product, defined by

\[
(f \land g)(x_1, \cdots, x_{n+m-1}) = \sum\_{\sigma} \text{sg} \sigma f(g(x_{\sigma(1)}, \cdots, x_{\sigma(m)}), x_{\sigma(m+1)}, \cdots, x_{\sigma(n+m-1)})
\]

where the sum extends over those permutations \( \sigma \) of \( \{1, \cdots, n+m-1\} \) for which \( \sigma(1) < \cdots < \sigma(m) \) and \( \sigma(m+1) < \cdots < \sigma(n+m-1) \). Its properties are formally completely analogous to those of \( f \bar{\sigma} g \), e.g. (3) holds; we define \( [f, g]^\circ \) as in (4); \( \mu \in C^\bullet(B, B) \) satisfies \( \mu \bar{\gamma} \mu = 0 \) (equivalent to \( [\mu, \mu]^\circ = 0 \) if char \( k \neq 2 \)) if and only if \( \mu \) defines a Lie algebra structure on \( B \), and the Chevalley-Eilenberg coboundary is given by \( \delta f = -[\mu, f]^\circ \). The product \( [f, g]^\circ \) is defined by

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\[ [f, g]^\sigma(x_1, \cdots, x_{n+m}) = \sum_{\sigma} s_{\sigma} \mu(f(x_{\sigma(1)}), \cdots, x_{\sigma(n)}), g(x_{\sigma(n+1)}, \cdots, x_{\sigma(n+m)}) \]

where the sum extends over those permutations \( \sigma \) of \( \{1, \cdots, n+m\} \) for which \( \sigma(1) < \cdots < \sigma(n) \) and \( \sigma(n+1) < \cdots < \sigma(n+m) \). The analogue of (7) holds, too. The references in the Lie case are [1], [6], [7]. Lemma 2.1 holds, too.

The case when \( \text{char } k = 2 \), or when \( k \) is a ground ring (unitary and commutative) in which division by 2 is not possible, needs separate treatment. Many details are as in [5]; we only comment on a few essentials. We set \( Q^0(f) = f \circ f \) (resp. \( f \wedge f \)) for \( n \) even, and \( Q^\sigma(f) = f \wedge f \) in the associative case, for \( n \) odd. In the Lie case we set \( Q^\sigma(f) \) equal to the sum on the right in (9), with \( f = g, m = n \) odd, and with the extra restriction \( \sigma(1) < \sigma(n+1) \). The GLA structures thus obtained are then strong in the sense of [5]. The operator \( Q^\sigma \) does not generally vanish on \( H^*(B, B) \), however, so the cup structure on \( H^*(B, B) \) is not abelian in the strong sense. This is not surprising in view of the fact that \( Q^\sigma(f) = Sq(f) \) when \( \text{char } k = 2 \) (cf. [3]).

3. Proof of the theorem. Lemma 2.1 proves the statement on the image of \( i^* \). The statements of Lemmas 3.1–4 show the existence of a GLA ("cup") structure on \( H^*(B, A/B) \). The homomorphism properties of \( i^* \) and \( \pi^* \) are obvious from (11); the homomorphism property of \( \delta^* \) is obvious from Lemma 3.4. All statements depend on the formal properties of \( \& \) and are made only for the associative case. In all cases \( A \) is an algebra, \( B \) a subalgebra; \( n = \deg f \) and \( m = \deg g \).

**Lemma 3.1.** Let \( A \) be an algebra. Then

\[ [f, g]^{\sigma} = [f, g]^{\sigma} + (-1)^n g \circ f + (-1)^{mn+m} f \circ g \]

defines a GLA structure on \( C^*(A, A) \). [When \( \text{char } k = 2 \), set \( Q(f) = Q^\sigma(f) = f \circ f \) for \( n \) odd and get a strong GLA structure.]

**Proof.** By tedious computation: as this lemma is not used in the following ones, the identities derived there (with \( B = A \)) may be used, in addition to those in \( \S 2 \). See also [9] for some further details on the operation (10).

**Lemma 3.2** Let \( f, g \in C^*(B, A/B) ; \) let \( \delta f = 0, \delta g = 0 \), and let \( \tilde{f}, \tilde{g} \in C^*(B, A) \) be such that \( f = \pi \circ \tilde{f} ; g = \pi \circ \tilde{g} \). Then \( \delta \tilde{f}, \delta \tilde{g} \) have values in \( B \), and

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belongs to $C^*(B, A)$ and its projection (by left-composition with $\pi$) on $C^*(B, A/B)$ depends on $\tilde{f}, \tilde{g}$ (given $f, g$) by no more than a coboundary.

**Proof.** Any two choices of $f$ differ by an element $\phi$ of $C^n(B, B)$. Hence, by (7)

$$[\tilde{f} + \phi, \tilde{g}] - [\tilde{f}, \tilde{g}] = [\phi, \tilde{g}] + (-1)^n g \delta \phi + (-1)^{m+n+1} f \delta \tilde{g}$$

$$= \delta \tilde{g} \delta \phi + (-1)^n g \delta \phi + (-1)^{m-1} \delta g \delta \phi$$

Two terms cancel; left composition with $\pi$ reduces the result to $(-1)^n \delta (g \delta \phi)$.

**Lemma 3.3.** Let $f, g, \tilde{f}, \tilde{g}$ be as in Lemma 3.2, and let $h \in C^{n-1}(B, A)$ be such that $\delta h = \tilde{f}$. Then $\pi \circ [\tilde{f}, \tilde{g}]$ is a coboundary in $C^*(B, A/B)$.

**Proof.** By computation

$$[\tilde{f}, \tilde{g}] = [\delta h, \tilde{g}] + (-1)^n h \delta \tilde{g} - (-1)^{m+n+1} h \delta \tilde{g}$$

$$= \delta [h, \delta g] + (-1)^{m-1} \delta h \delta \tilde{g} + (-1)^{m+1} \delta h \delta \tilde{g}$$

Two terms cancel, one is zero, and the rest are coboundaries. Left-composition with $\pi$ yields coboundaries in $C^*(B, A/B)$.

**Lemma 3.4.** Let $f, g, \tilde{f}, \tilde{g}$ be as in Lemma 3.2; then

$$\delta [\tilde{f}, \tilde{g}] = [\delta \tilde{f}, \delta \tilde{g}] \in C^*(B, B).$$

**Proof.** By computation:

$$\delta [\tilde{f}, \tilde{g}] = \delta [\tilde{f}, \tilde{g}] + (-1)^n \delta (g \delta \tilde{f}) + (-1)^{m+n+1} \delta (\tilde{f} \delta \tilde{g})$$

$$= \delta [\tilde{f}, \tilde{g}] + (-1)^n \{(-1)^{n+1} [\delta \tilde{f}, \tilde{g}] + (-1)^n \delta \tilde{g} \delta \tilde{f} + \tilde{g} \delta \delta \tilde{f}\}$$

$$+ (-1)^{m+n+1} \{(-1)^{m+1} [\delta \tilde{g}, \tilde{f}] + (-1)^m \delta \tilde{f} \delta \tilde{g} + \tilde{f} \delta \delta \tilde{g}\}$$

$$= \delta [\tilde{f}, \tilde{g}] + [\delta \tilde{f}, \delta \tilde{g}] + \tilde{g} \delta \tilde{f} + (-1)^{m+n+1} \delta \tilde{g} \delta \tilde{f}$$

**References**


UNIVERSITY OF PENNSYLVANIA

**BOUNDED APPROXIMATION BY POLYNOMIALS WITH RESTRICTED ZEROS**

**BY CHARLES KAM-TAI CHUI**

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1. **Introduction.** Let $C$ be a rectifiable Jordan curve, $D$ its interior. A sequence of polynomials $P_n(z)$ is said to converge boundedly to a function $f(z)$ in $D$, or equivalently, $f(z)$ is said to be boundedly approximated by the polynomials $P_n(z)$ in $D$, if $\sup \{ |P_n(z)| : z \in D \}$ is bounded as a function of $n$, and $\{ P_n(z) \}$ converges to $f(z)$ throughout $D$. It is known [1], [6] that $f(z)$ can be boundedly approximated by polynomials in $D$ if and only if $f(z)$ is a bounded holomorphic function in $D$. In this paper we consider the more delicate bounded approximation problem in which the zeros of the polynomials are required to lie on the boundary $C$. Polynomials whose zeros lie on $C$ are called $C$-polynomials.

A different kind of approximation by $C$-polynomials was studied by G. R. MacLane [5]. He proved that if $f(z)$ is holomorphic and zero free in $D$, then there exists a sequence of $C$-polynomials which converges to $f(z)$ uniformly on every compact subset of $D$. This result was later extended by J. Korevaar [3] and his students [4] to more general sets $D$.

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