ON DIRECT PRODUCTS OF GENERALIZED
SOLVABLE GROUPS

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Let \( G_\alpha (\alpha \in \Gamma) \) be a set of groups. The direct product \( \prod \{ G_\alpha | \alpha \in \Gamma \} \)
is the set of all functions \( f \) on \( \Gamma \) such that \( f(\alpha) \in G_\alpha \) for all \( \alpha \in \Gamma \), with
multiplication of functions defined componentwise. The direct sum
\( \sum \{ G_\alpha | \alpha \in \Gamma \} \) is the subgroup of \( \prod \{ G_\alpha | \alpha \in \Gamma \} \) consisting of all
functions \( f \) with finite support.

A collection \( \mathcal{G} \) of groups is called a class of groups if \( E \in \mathcal{G} \), and iso-
morphic images of \( \mathcal{G} \) groups are \( \mathcal{G} \) groups. We use the following nota-
tion of P. Hall [1]. If \( \mathcal{G} \) is a class of groups, \( S(\mathcal{G}) \), \( Q(\mathcal{G}) \), \( DS(\mathcal{G}) \),
\( DP(\mathcal{G}) \) denote respectively the classes of groups which are subgroups,
quotient groups, direct sums and direct products of \( \mathcal{G} \) groups.

The following theorem was proved by Merzulakov in [2].

**Theorem 1.** If \( \mathcal{G} \) is a class of groups satisfying
(a) \( S(\mathcal{G}) = \mathcal{G} \),
(b) \( Q(\mathcal{G}) = \mathcal{G} \),
(c) \( G \) is a finite \( \mathcal{G} \) group if and only if \( G \) is nilpotent, then \( DP(\mathcal{G}) \neq \mathcal{G} \).

In this paper, a similar theorem is obtained for generalized solvable
groups. Before stating these results, we need several definitions.

**Definition 1.** Let \( G \) be a group, \( x \in G \), \( g \in G \). Define \([g, 0x] = g\),
and inductively \([g, nx] = [\{g, (n-1)x\}, x] \) for each positive integer \( n \).
\( x \) is called a left \( G \) Engel element if for each \( g \in G \) there exists an inte-
ger \( n = n(g) \) such that \([g, nx] = e\).

The Hirsch-Plotkin radical of a group \( G \) is the maximum normal
locally nilpotent subgroup of \( G \). We denote the Hirsch-Plotkin radical
of \( G \) by \( \phi_1(G) \).

**Definition 2.** Let \( G \) be a group and \( \phi_0(G) = E \). If \( \alpha \) is not a limit
ordinal, define \( \phi_\alpha(G) \) by \( \phi_\alpha(G)/\phi_{\alpha-1}(G) = \phi_1(G/\phi_{\alpha-1}(G)) \). If \( \alpha \) is a limit
ordinal, define \( \phi_\alpha(G) \) by \( \phi_\alpha(G) = \bigcup \{ \phi_\beta | \beta < \alpha \} \). If for some ordinal \( \sigma \),
\( \phi_\sigma(G) = G \), \( G \) is called an \( LN \)-radical group.

In the following, \( \mathcal{L} \) will denote the class of \( LN \)-radical groups. If
\( G \in \mathcal{L} \), and \( \sigma \) is the least ordinal for which \( \phi_\sigma(G) = G \), \( \sigma \) is called the
radical class of \( G \). It is well known that \( S(\mathcal{L}) = \mathcal{L} \), \( Q(\mathcal{L}) = \mathcal{L} \), and
that every solvable group is in \( \mathcal{L} \) [3]. It is easily shown that if \( n \) is a
positive integer, there exist finite solvable groups of radical class \( n \)
[4, p. 220].

We need the following theorem of Plotkin [3].

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Theorem 2. If \( G \in \mathcal{L} \), then the set of left Engel elements of \( G \) is a subgroup, and this subgroup coincides with the Hirsch-Plotkin radical of \( G \).

In the remainder of this paper, \( J \) will denote the set of nonnegative integers.

Theorem 3. Let \( n \in J \) and \( G_n \in \mathcal{L} \) have radical class \( n \). Then \( G = \prod \{ G_n | n \in J \} \in \mathcal{L} \).

Proof. Let \( R_k = \prod \{ \phi_k(G_n) | n \in J \} \) and \( R = \bigcup \{ R_k | k \in J \} \). Then \( R \triangleleft G \) and \( R \neq G \). We show that \( \phi_1(G/R) = E \).

Suppose to the contrary that \( \phi_1(G/R) \neq E \) and let \( y \in R \). Then \( y \) is a left \( G/R \) Engel element. Thus for each \( x \in G \setminus R \), there exists a positive integer \( n = n(x) \) such that \([x, ny] \in R \).

Hence for each \( x \in G \setminus R \), there exist nonnegative integers \( n = n(x) \) and \( k = k(x) \) such that \([x, ny]^k \in R_k \).

We now construct an \( x \in G \) for which the above assertions do not hold. Since \( y \in R \), there exists \( i \in J \) such that \( y(i) \in \phi_1(G_i) \). By Theorem 2, \( y(i) \) is not a left \( G_i \) Engel element. Hence there exists \( x_{i_1} \in G_{i_1} \) such that \([x, sy(i)] \in \phi_0(G_{i_1}) = E \) for all \( s \in J \).

Suppose nonnegative integers \( i_1 < i_2 < \cdots < i_r \) and elements \( x_{i_j} \in G_{i_j} (1 \leq j \leq r) \) have been found so that for \( 1 \leq j \leq r \), \([x, sy(i_j)] \in \phi_{j-1}(G_{i_j}) \) for all \( s \in J \). Since \( y \in R \), there exists an integer \( i_{r+1} > i_r \) such that \( y(i_{r+1}) \in \phi_{r+1}(G_{i_{r+1}}) \). Thus, by Theorem 2 \( y(i_{r+1}) \) is not a left \( G_{i_{r+1}} \) Engel element. Hence there exists \( x_{i_{r+1}} \in G_{i_{r+1}} \) such that \([x, sy(i_{r+1})] \in \phi_r(G_{i_{r+1}}) \) for all \( s \in J \).

Let \( I = \{ i_1, i_2, \cdots, i_r, \cdots \} \). Define \( x \in G \) as follows: \( x(\eta) = x_\eta \) if \( \eta \in I \) and \( x(\eta) = e \) otherwise. Let \( k \in J \). Then \([x, sy] \in R_k \) for all \( s \in J \).

This is contrary to the first paragraph of this proof.

Theorem 4. Let \( \mathfrak{B} \) be a class of groups such that

(a) \( \mathfrak{B} \subseteq \mathcal{L} \),

(b) every finite solvable group is contained in \( \mathfrak{B} \).

Then \( DP(\mathfrak{B}) \neq \mathfrak{B} \).

Proof. The proof follows from Theorem 3 and the existence of finite solvable groups of radical class \( n \) for each \( n \in J \).

The direct product \( \prod \{ G_\alpha | \alpha \in \Gamma \} \) is called a direct power of \( H \) if each \( G_\alpha \) is isomorphic to \( H \). If \( \mathfrak{B} \) is a class of groups, \( dp(\mathfrak{B}) \) will denote the class of groups which are direct powers of \( \mathfrak{B} \) groups.

In the next theorem, \( S \) will denote the class of solvable groups.

Theorem 5. If \( \mathfrak{B} \) is a class of groups such that

(a) \( \mathfrak{B} \subseteq \mathcal{L} \),

(b) \( DS(\mathfrak{B}) \subseteq \mathfrak{B} \),

Then \( dp(\mathfrak{B}) \neq \mathfrak{B} \).
PROOF. Let \( G = \sum \{ G_n \, | \, n \in J \} \) where \( G_n \) is solvable of radical class \( n \). Then \( G \in \Phi \) and has radical class \( \omega \). Let \( H = \prod \{ H_k \, | \, k \in J, H_k \cong G \} \). \( H \) has a subgroup satisfying the hypothesis of Theorem 3. Hence \( H \in \Phi \). Consequently, \( H \in \Phi \).

Classes of groups satisfying the conditions of Theorems 4 and 5 include the classes \( SN^* \), \( SI^* \), subsolvable and polycyclic.

BIBLIOGRAPHY


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ALGEBRAIZATION OF ITERATED INTEGRATION ALONG PATHS

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If \( \Omega \) is the vector space of \( C^\infty \) 1-forms on a \( C^\infty \) manifold \( M \), then iterated integrals along a piecewise smooth path \( \alpha : [0, 1] \to M \) can be inductively defined as below:

For \( r \geq 2 \) and \( w_1, w_2, \cdots, \in \Omega \),

\[
\int \alpha w_1 \cdots w_r = \int_0^1 \left( \int \alpha^t \left( \int \alpha^{t+1} \cdots \int \alpha^{r+s} \right) \right) dt
\]

where \( \alpha^t = \alpha | [0,t] \). (See [3].)

This note is based on the following algebraic properties of the iterated integration:

(a) \( (\int \alpha w_1 \cdots w_r)(\int \alpha w_{r+1} \cdots w_{r+s}) = \sum \int \alpha w_1 \cdots w_{r+s} \) summing over all \((r,s)\)-shuffles, i.e. those permutations \( \sigma \) of \( \{1, \cdots, r+s\} \) with \( \sigma^{-1}(1) < \cdots < \sigma^{-1}(r), \sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s) \).

(b) If \( p = \alpha(0) \) and if \( f \) is any \( C^\infty \) function on \( M \), then

\[
\int \alpha f w = \int \alpha (df) w + f(p) \int \alpha w.
\]

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