INTERMEDIATE EXTENSIONS IN $L^p$

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1. Introduction. Let $A(x, D)$ be an elliptic operator defined on Euclidean $n$-dimensional space and let $q(x)$ be a locally square integrable function. Let $A_0$ and $B_0$ denote the operators $A(x, D)$ and $A(x, D) + q(x)$ acting on the set $C_0^\infty$ of infinitely differentiable functions, respectively. Under suitable regularity conditions on the coefficients of $A(x, D)$ the minimal and maximal closed extension of $A_0$ in $L^p$ coincide for $1 < p < \infty$. Without further restrictions on $q$, this is not true for $B_0$.

The purpose of the present investigation is to find sufficient conditions on $q$ such that some closed extension of $B_0$ will have the same essential spectrum as the closure $A$ of $A_0$. For $p = 2$ we found it convenient in [11] to employ regularly accretive extensions introduced by Kato [13]. However, this theory employs Hilbert space structure and is unapplicable for $p \neq 2$. Moreover, some of the $L^2$ estimates employed in [11] have no known counterparts in $L^p$ for $p \neq 2$.

Our approach has been to develop a theory of extensions in Banach space which generalizes Kato's development. We call such operators "intermediate extensions." Under suitable conditions on $q(x)$ we are able to show that these extensions have the desired properties.

2. Intermediate extensions. Let $A_0$ be a densely defined, preclosed linear operator from a Banach space $X$ to a Banach space $Y$. Then $D(A_0^*)$ is weakly* dense in $Y^*$. Let $S$ be a linear manifold in $D(A_0^*)$ which is also weakly* dense in $Y^*$. We consider all closed extensions $A$ of $A_0$ such that $D(A^*) \supseteq S$. The closure $\overline{A}$ of $A_0$ is the smallest such extension and therefore will be called the minimal extension of $A_0$. There is a largest such extension $\overline{A}$. $D(\overline{A})$ consists of those $u \in X$ for which there is an $f \in Y$ satisfying

$$(u, A_0^*v) = (f, v) \quad \text{for all } v \in S.$$  

We then set $\overline{A} u = f$. This operator is well defined, for if $(u, A_0^*v) = (g, v)$ for all $v \in S$, then $(f - g, v) = 0$ for all such $v$. Since $S$ is weakly* dense in $Y^*$, we have $f = g$. Moreover, if $A$ is any closed extension of $A_0$ with $D(A^*) \supseteq S$, then

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\[(u, A^*v) = (Au, v) \quad \text{for all } u \in D(A), \ v \in D(A^*).\]

In particular, this holds for \(v \in S\), showing that \(u \in D(\overline{A})\) and \(\overline{A}u = Au\). We call \(\overline{A}\) the maximal extension of \(A_0\) relative to \(S\).

We now consider a method of obtaining other closed extensions of \(A_0\) "between" the minimal and maximal extensions. We call such extensions intermediate. Let \(W\) be a Banach space containing \(D(A_0)\) and continuously embedded in \(X\). Similarly, let \(Z\) be a Banach space containing \(S\) and continuously embedded in \(Y^*\).

**Definition 2.1.** An operator \(A\) from \(X\) to \(Y\) will be called an intermediate extension of \(A_0\) relative to \(W\) and \(Z\) (or briefly a \(W-Z\) extension of \(A_0\)) if

(a) There is a continuous bilinear form \(a(u, v)\) on \(W \times Z\) such that \(a(u, v) = (A_0u, v)\) on \(D(A_0) \times Z\). 

(b) \(u \in D(A)\) and \(Au = f\) if and only if \(u \in W\), \(f \in Y\) and \(a(u, v) = (f, v)\) for all \(v \in Z\).

Note that a \(W-Z\) extension need not be closed.

**Lemma 2.2.** If \(D(A_0)\) is dense in \(W\), then a necessary and sufficient condition that \(A_0\) have a unique \(W-Z\) extension is that

\[(2.1) \quad |(A_0u, v)| \leq C|u|_W|v|_Z \quad \text{for all } u \in D(A_0), \ v \in Z.\]

**Remark 2.3.** If \(Z = Y^*\) and \(W = D(\overline{A})\), then the \(W-Z\) extension \(A\) of \(A_0\) is \(\overline{A}\), the minimal extension. If \(W = X\) and \(Z\) is the closure of \(S\) in \(D(A^*_0)\), then \(A = \overline{A}\), the maximal extension.

**Theorem 2.4.** A necessary and sufficient condition that a \(W-Z\) extension \(A\) be closed is that

\[(2.2) \quad \|u\|_W \leq C(\|Au\|_Y + \|u\|_X), \quad u \in D(A).\]

**Theorem 2.5.** If

\[(2.3) \quad \|u\|_W \leq C(\sup_{v \in Z} |(A_0u, v)| /\|v\|_Z + \|u\|_X), \quad u \in D(A_0),\]

then the \(W-Z\) extension \(A\) of \(A_0\) is closed.

The class \(\Phi(X, Y)\) of Fredholm operators from \(X\) to \(Y\) is defined as the set of densely defined, closed, linear operators from \(X\) to \(Y\), having closed ranges with the dimensions of their null spaces and codimensions of their ranges both finite (cf. [1]). For \(A \in \Phi(X, Y)\), the index \(i(A)\) of \(A\) is the difference between the dimension of its null space and the codimension of its range.

**Theorem 2.6.** A \(W-Z\) extension \(A\) is in \(\Phi(X, Y)\) if and only if
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(2.4) \[ \| u \|_W \leq C \| A u \|_X + \| u \|_K, \quad u \in D(A) \]

and

(2.5) \[ \| v \|_Z \leq C \| A^* v \|_{X^*} + \| v \|_{K'}, \quad v \in D(A^*), \]

where the norm $\| \cdot \|_K$ (resp. $\| \cdot \|_{K'}$) is completely continuous with respect to the norm of $W$ (resp. $Z$).

3. Definitions and main results. We shall be working in the $H^{s,p}$ spaces for $s$ real and $1 < p < \infty$. Let $C^\infty_0$ denote the set of infinitely differentiable complex valued functions with compact supports in $n$-dimensional Euclidean space $E^n$. For $\phi \in C^\infty_0$ we define the norm

(3.1) \[ \| \phi \|_{s,p} = \left[ \int \left| \mathcal{F}^{-1} (1 + |\xi|^2)^{s/2} \mathcal{F} \phi \right|^p dx \right]^{1/p}, \]

where $\mathcal{F}$ denotes the Fourier transform, $x = (x_1, \cdots, x_n)$ is the variable in $E^n$, $\xi = (\xi_1, \cdots, \xi_n)$ is the argument of $\mathcal{F}$ and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. For $s$ a nonnegative integer, the norm (3.1) is equivalent to the sum of the $L^p$ norms of $\phi$ and all its derivatives up to order $s$. The completion of $C^\infty_0$ with respect to the norm (3.1) is denoted by $H^{s,p}$. Let $q(x)$ be a measurable function on $E^n$. For $\alpha$ real and $1 < p < \infty$ we set

\[ M_{a,p}(q) = \sup_x \int_{|x-y|<1} |q(y)|^p dy, \quad \alpha > 0, \]

\[ = \sup_x \int_{|x-y|<1} |q(y)|^p (1 - \log |x - y|) dy, \quad \alpha = 0, \]

\[ = \sup_x \int_{|x-y|<1} |q(y)|^p |x - y|^\alpha dy, \quad -n < \alpha < 0, \]

\[ = \sup_x |q(x)|, \quad \alpha \leq -n. \]

We let $M_{a,p}$ denote the set of those measurable $q$ for which $M_{a,p}(q) < \infty$.

We shall consider an elliptic operator

(3.2) \[ A(x,D) = \sum_{|\mu| \leq m} a_\mu(x) D^\mu \]

of order $m$ on $E^n$, where $\mu = (\mu_1, \cdots, \mu_n)$ is a multi-index of non-negative integers, $|\mu| = \mu_1 + \cdots + \mu_n$, and
We assume that \( A(x, D) \) is uniformly elliptic in \( E^n \). This means that there is a constant \( C_0 > 0 \) such that
\[
\left| \sum_{|\mu| = m} a_\mu(x) \xi^\mu \right| \geq C_0 |\xi|^m
\]
for all points \( x \in E^n \) and real vectors \( \xi \), where
\[
\xi^\mu = \xi_1^\mu \cdots \xi_n^\mu.
\]
The formal adjoint of \( A(x, D) \) is given by
\[
A'(x, D) = \sum_{|\mu| \leq m} D^\mu(a_\mu(x)).
\]
We assume that the coefficients of both \( A(x, D) \) and \( A'(x, D) \) are bounded and that for \( |x| \to \infty \) the \( a_\mu(x) \) are continuous. All of the coefficients of both \( A(x, D) \) and \( A'(x, D) \) are to approach limits as \( |x| \to \infty \).

Let \( A_0 \) denote \( A(x, D) \) acting on \( C_0^\infty \) and considered as an operator on \( L^p \). As is well known, \( A_0 \) is preclosed and all closed extensions having \( C_0^\infty \) in the domains of their adjoints coincide (cf., e.g., [2]). Set
\[
A(\infty, D) = \sum_{|\mu| \leq m} a_\mu D^\mu,
\]
where \( a_\mu \) is the limit of \( a_\mu(x) \) as \( |x| \to \infty \).

Let \( A \) and \( A_\infty \) denote the closed extensions of \( A_0 \) and \( A(\infty, D) \) acting on \( C_0^\infty \), respectively. The spectrum \( \sigma(A_\infty) \) of \( A_\infty \) can be computed by means of Fourier transforms (cf., e.g., [3]). It is the set \( R_0 \) of values assumed by
\[
A(\infty, \xi) = \sum_{|\mu| \leq m} a_\mu \xi^\mu
\]
where \( \xi \) ranges over all real values. For \( \lambda \in R_0 \) the range of \( A_\infty - \lambda \) is not closed in \( L^p \).

For an arbitrary operator \( E \) in a Banach space \( X \) the \( \Phi \)-set \( \Phi_E \) of \( E \) is defined as the set of those complex \( \lambda \) for which \( E - \lambda \in \Phi(X) \equiv \Phi(X, X) \). In the case of \( A_\infty \) we have
\[
(3.3) \quad \Phi_{A_\infty} = \rho(A_\infty) = CR_0,
\]
where \( CR_0 \) denotes the complement of \( R_0 \) in the complex plane. For the operator \( A \) we have
Theorem 3.1.

\[ \Phi_A = CR_0. \]

According to Wolf [4] the essential spectrum \( \sigma_{\text{ew}}(E) \) of an operator \( E \) is defined as \( C\Phi_B \). Thus another way of stating Theorem 3.1 is

\[ \sigma_{\text{ew}}(A) = \sigma_{\text{ew}}(A_\infty) = \sigma(A_\infty) = R_0. \]

Let \( s \) be a fixed real number satisfying \( 0 \leq s \leq m \). We assume that \( q(x) \) is a function defined on \( E^n \) such that

\[ q_1 \in M_{n, \beta}, \quad \alpha < \beta s - n, \]
\[ q_2 \in M_{n, \alpha}, \quad \beta < \beta'(m - s) - n, \quad \beta' = \beta'(p - 1), \]

\[ \int_{|x - y| < 1} |q(y)| \, dy \to 0 \quad \text{as } |x| \to \infty. \]

The operator \( A(x, D) + q \) acting on \( C_0^\infty \) is preclosed in \( L^p \). Denote it by \( B_0 \).

Theorem 3.2. The operator \( B_0 \) has an intermediate extension \( \beta \) relative to the spaces \( H^s \) and \( H^{m-s} \).

The extension \( \beta \) will be called the \( s \)-extension of \( B_0 \).

Theorem 3.3. Under the above hypotheses,

\[ \Phi_B \supseteq \Phi_A \]

and if \( \lambda \in \Phi_A \), then

\[ i(B - \lambda) = i(A - \lambda). \]

Theorem 3.4. If, in addition, \( CR_0 \) is not empty, then

\[ \Phi_B = \Phi_A \]

and (3.6) holds.

Another definition of essential spectrum given in [12] adds to \( \sigma_{\text{ew}}(E) \) all points \( \lambda \) of \( \Phi_B \) for which \( i(E - \lambda) \neq 0 \). We denote the essential spectrum according to this definition by \( \sigma_{\text{em}}(E) \). Thus by Theorems 3.3 and 3.4 we have

\[ \sigma_{\text{ew}}(B) \subseteq \sigma_{\text{ew}}(A), \quad \sigma_{\text{em}}(B) \subseteq \sigma_{\text{em}}(A) \]

in general, and

\[ \sigma_{\text{ew}}(B) = \sigma_{\text{ew}}(A), \quad \sigma_{\text{em}}(B) = \sigma_{\text{em}}(A) \]

when \( \sigma_{\text{ew}}(A) \) is not the whole complex plane.
We now consider the perturbation of \( A(x, D) \) by an operator
\[
C(x, D) = \sum_{|\mu| < m} C_\mu(x) D^\mu.
\]
For \(|\mu| \leq s\) we assume that \( C_\mu(x) = C_\mu(x) C_\mu(x) \), where
\[
C_\mu \in M_{\alpha-p|\mu|, p}, \quad C_\mu \in M_{\beta, p'},
\]
where \( \alpha < p - n \) and \( \beta < p'(m-s) - n \). For \(|\mu| > s\) we assume
\[
G_{|\mu|} C_\mu \in M_{\beta, p'},
\]
where the operator \( G_t \) is given by
\[
G_t = 3^{-1}(1 + |\xi|^2) 3^t 0v.
\]
We also assume
\[
\int_{|x-y| < 1} |C_\mu(y)| \, dy \to 0 \quad \text{as} \quad |x| \to \infty, \quad |\mu| \leq s,
\]
\[
\int_{|x-y| < 1} |G_{|\mu|} C_\mu(y)| \, dy \to 0 \quad \text{as} \quad |x| \to \infty, \quad |\mu| > s.
\]

**Theorem 3.5.** Let \( L_0 \) denote the operator \( A(x, D) + C(x, D) \) acting on \( C_0^\infty \). Under the above hypotheses, it has an \( s \)-extension \( L \). Moreover we have
\[
\Phi_L \supseteq \Phi_A
\]
and if \( \lambda \in \Phi_A \), \( \delta(L - \lambda) = \delta(A - \lambda) \). If \( CR_0 \) is not empty, then \( \Phi_L = \Phi_A \).

**Bibliography**

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ON SELF DUAL L C A GROUPS

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DEFINITION 1.1. Let $G$ be a locally compact Hausdorff Abelian group with a character group $G^*$. (Hereafter we shall call such groups $G$ as L C A groups.) $G$ is called self dual if there is a topological isomorphism $T: G^* \rightarrow G$ from $G$ onto $G^*$. Some examples of self dual groups are known in literature, but the structure of all such groups is an open problem (see page 423 of [1]). In this note we announce the structure of those self dual groups which are torsion free as abstract Abelian groups. We state some definitions before announcing the main theorem. The complete details will appear elsewhere.

DEFINITION 1.2. Let $J$ be an index set. Let $G_\alpha$ be a L C A group for each $\alpha \in J$. Let $H_\alpha \subset G_\alpha$ be a compact, open subgroup of $G_\alpha$ for every $\alpha \in J$. By the local direct sum $G$ of the groups $G_\alpha$ modulo $H$, we mean the subgroup of $\prod_{\alpha \in J} G_\alpha$ consisting of those elements for which all but a finite number of coordinates lie in $H_\alpha$. Notice that $H = \prod_{\alpha \in J} H_\alpha$ is contained in $G$. We topologise $G$ in such a way that $H$ is declared to be open in $G$, and the relative topology on $H$ as a subspace of $G$ coincides with the product topology of the spaces $H_\alpha$ where $H_\alpha$ is given the relative topology from $G_\alpha$. We write $G = \sum_{\alpha \in J} G_\alpha$. With this definition $G$ is also an L C A group.

DEFINITION 1.3. Let $p$ be a prime integer $>0$. Then $J_p$ denotes the field of $p$-adic numbers with usual addition and topology. With this addition and topology, $J_p$ is a locally compact Abelian group. Any compact, open subgroup $H_\alpha$ of $J_p$ is called $p$-adic integers. A local direct sum $\sum_{\alpha \in X} G_\alpha$ of the L C A groups $G_\alpha$ is called a canonical $p$-group if each $G_\alpha$, where $\alpha \in X$, is isomorphic to $p$-adic numbers and, in each $G_\alpha$, some compact open subgroup is fixed in advance.

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