BOOK REVIEW

Real and complex analysis. By Walter Rudin. McGraw-Hill, New York, 1966. xi+412 pp.

The book under review, *Real and complex analysis* by Walter Rudin, is excellent and without doubt represents a valuable contribution to the mathematical literature.

This book excels primarily in two important respects. The first is that the choice of topics serves as a superior introduction into much of what is current in analysis, in particular to the branches of harmonic analysis, partial differential equations, several complex variables, and Banach algebras. The second is that it blends both the concrete and abstract viewpoints and tends to do away with the notion that prevailed in the past separating analysis into "soft" analysis versus "hard" analysis.

The book on one hand establishes the Riesz representation theorem for locally compact Hausdorff spaces and on the other hand establishes the Denjoy-Carleman theorem for quasi-analytic functions, the former theorem being a good example of what was called "soft" and the latter of what was called "hard." It also contains a section entitled "An Abstract Approach to the Poisson Integral" (§5.22) which brings this point to the fore even more.

The book also has several other attributes which make it excellent. One of these is that for a number of nontrivial theorems the book is self-contained—needless to say, not a common occurrence.

For example, take the Denjoy-Carleman theorem mentioned above. To prove it, one needs the 2nd Paley-Wiener theorem, a discussion concerning entire functions, and the Plancherel theorem. All these pertinent facts can be found discussed previously in the text.

The book is replete with many other examples of this nature. A few illustrations are the following: the Beurling invariant subspace theorem, Mergelyan's theorem, and the Riesz L^p theorem for conjugate harmonic functions.

Another outstanding attribute of the book is that it is both well written and correctly written. During the academic year 1966–67, the reviewer used the book as the text for two different graduate courses at the University of California at Riverside. The errors and misprints that the reviewer found were remarkably few and far between.

One of the graduate courses taught was the typical one in real analysis given at the first year graduate level throughout the country. The other was a second year graduate course devoted to harmonic analysis with applications to partial differential equations—in particular the heat equation. Rudin's book provided the basis for the first part of this course. By a proper selection from various chapters in the book, the reviewer was able to find the material that was necessary to provide both an interesting introductory course in harmonic analysis and background material for the study of the heat equation.

This is once again best illustrated by example. The pertinent theorem is the Denjoy-Carleman theorem on quasi-analytic functions. As was mentioned previously, this theorem is best proved through the use of several theorems in harmonic analysis, all covered quite adequately in Rudin's book. After the theorem on quasi-analytic functions is established, one can pass to the study of the heat equation (not found in Rudin's book) where one rather quickly establishes the classical theorem of Tychonoff concerning the uniqueness of the heat equation. One of the best ways to show that the $\exp(\alpha x^2)$ estimate in Tychonoff's theorem cannot be replaced by $\exp(\alpha x^{2+\epsilon})$ for $\epsilon > 0$ is to use the Denjoy-Carleman theorem on quasi-analytic functions, and this is precisely what the reviewer did. So one sees that there are indeed ways the book under review can be used as a second year graduate text.

However, it is more relevant to discuss the effect of Rudin's book on the graduate course given at the first year level. Here, it is quite good also. Presupposing that the student is reasonably well prepared, the book sails right into the relevant concepts of measure theory and real analysis without spending an undue amount of time on logical and set-theoretic preliminaries, these latter facts being introduced as the occasion for their need arises.

The book is well written, but it is closely written. Consequently, the instructor will still find plenty to do in filling in the details of a number of proofs, especially for first year graduate students.

The first nine chapters of the book contain roughly speaking, the amount of material covered in the first year graduate course mentioned earlier given by the reviewer. And, at that, a number of topics were omitted. Nevertheless, in the reviewer's opinion, this constituted a good first year course in real analysis. It is possible to cover more ground in one year and thus in one year to teach a joint course in real and complex analysis out of Rudin's book, but this probably can only be done at the expense of omitting even more topics in real analysis and, of course, omitting topics in complex analysis which should quite probably not be omitted, e.g., an asymptotic development of Stirling's formula, using contour integration as is done, say, in Ahlfor's book on complex analysis. Naturally, one cannot cover in one year what normally takes two years no matter how efficiently the topics are arranged. Nevertheless, it appears that Rudin's book, especially in the later chapters, accomplishes what Rudin asserted in his preface that he wished to do, namely, to show the subjects of "real variables" and "complex variables" are inextricably combined and do not constitute two distinct subjects.

In short, it is the reviewer's opinion that Rudin's book can be used either as a text for the standard first year graduate course in real analysis or as the text for a combined one year course in real and complex analysis. But in the latter case something will have to give.

The main criticism with the use of Rudin's book as a text for a graduate course lies in the lack of coordination of the problems at the end of each chapter with the material in the chapter. One would expect that the ordering of the problems would have some correspondence with the manner in which the material is presented in a given chapter. Unfortunately, this is not the case. (For example, look at Exercise 2 of Chapter 1 which deals in part with Fatou's Lemma. Compare it with where Fatou's Lemma occurs in Chapter 1.)

Also, a number of the exercises in the book are difficult and some are even wrong. (This fact may not be considered a disadvantage by certain individuals.) For example, Exercise 3 on page 176 is incorrect, and Exercise 6 on the same page, after the misprint is corrected, should be reworded to be made correct. It is incorrect as stated.

However, the drawback of the problem sections is minor compared to the remarkable total effectiveness of the book as a text. Also, the very good students enjoy the challenge of the more difficult problems. Furthermore the problem sections are quite extensive and interesting.

A chapter by chapter summary of the book will now be given. Chapter 1 deals with abstract integration and measure theory. The concept of a measurable space is developed by analogy with the concept of a topological space. The notion of the integral is introduced, and the fundamental theorems of integration theory are proved.

In Chapter 2, the Riesz representation theorem for positive linear functionals on a locally compact Hausdorff space is established, and the connection between linear functionals and measures is exploited to develop a number of the fundamental concepts of measure theory.

In Chapter 3, Jensen's inequality is proved and the basic inequalities of L^p-spaces are established.

Chapter 4 establishes the basic concepts of elementary Hilbert space theory and introduces the reader to trigonometric series. The Riesz-Fischer theorem and Parseval's theorem are proved.

Chapter 5 establishes several theorems in Banach space theory and then applies these theorems to obtain results in Fourier analysis, e.g., to show there exists a two-way infinite sequence tending to zero in both directions which is not a sequence of Fourier coefficients.

Chapter 6 continues the development of the connection between measure theory and linear functionals, e.g., the Von Neuman proof of the Radon-Nikodym theorem using bounded linear functionals on Hilbert space can be found here.

Chapter 7 establishes Fubini's theorem and shows by way of three different examples the relevance of the hypotheses in the theorem.

Chapter 8 is devoted to the connection between measure theory and functions of bounded variation. Here Lebesgue's classical theorem on the derivative of an indefinite integral in Euclidean k-space is established.

In Chapter 9, the elements of Fourier analysis on the real line are developed, Plancherel's theorem is proved, and the concept of L^1 as a Banach algebra is introduced.

Chapter 10 develops the standard elementary theory for holomorphic functions, and Chapter 11 does the same for harmonic functions.

Chapter 12 integrates a number of the concepts in the previous chapters and uses Phragmen-Lindelöf methods to establish the three-lines theorem. Also using the techniques of Calderón and Zygmund, the Hausdorff-Young theorem (the theorem which relates the L^q-norm of a Fourier transform to the L^p-norm of its function where $1 \leqq p \leqq 2$) is established.

Chapter 13 is devoted to Runge's theorem, Cauchy's theorem, and related topics. The Hahn-Banach theorem and the Riesz representation theorem are used to prove Runge's theorem.

Chapter 14 deals with conformal mapping and the Riemann mapping theorem. Chapter 15 is devoted to entire functions, Blashke products, and closes with the Müntz-Szasz theorem.

Chapter 16 is concerned with the Hadamard gap theorem, the Monodromy theorem, and the little Picard theorem.

Chapter 17 is devoted to the study of the Hardy classes of functions defined in the interior of the unit disc. In particular, the classical F. and M. Riesz theorem is established. Also Beurling's invariant subspace theorem is proved, and the connection between the L^p-norms of conjugate harmonic functions is established.

Chapter 18 is devoted to the elementary theory of Banach algebras and some classical applications (namely to Wiener's theorem on the inversion of absolutely convergent Fourier series).

Chapter 19 is devoted to the two classical Paley-Wiener theorems and to the theory of quasi-analytic functions, in particular the Denjoy-Carleman theorem. For those people who have experienced reading Wiener's book on *Fourier transforms in the complex domain*, this chapter by itself makes the entire book worthwhile.

Chapter 20 is essentially devoted to a proof of Mergelyan's theorem.

The Notes and Comments section is interesting, pointing out alternative proofs, various historical facts, and further references. There is also an appendix and a prologue.

The following is a list of misprints: Page 29, line 7; page 37, line 6–; page 48, line 9, page 102, line 11; page 103, line 4–; page 104, line 6–; page 226, line 4–; page 291, line 8; page 303, line 5–; page 339, line 19; page 370, line 10; page 376, line 4; page 377, line 10; and page 398, line 11.

On page 145, lines 25 and 26, the material appears to be incorrect as stated, but can easily be corrected. On page 226, the inequality in line 17 is incorrect.

There are probably other misprints and incorrect statements in the book, but it appears to the reviewer that, considering everything, the total number of errors and misprints is remarkably small indeed.

In summary, we close with the opinion shared by the reviewer, his colleagues, and a number of other analysts throughout the country. Rudin's book catches much of the current spirit in analysis; it is an excellent introduction to the subject; it is an outstanding book.

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