ON THE GEOMETRIC THEORY OF FUNCTIONS MEROMORPHIC IN A DISC[1]

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Let $w=f(z)$ be a nonconstant meromorphic function defined in the open unit disc D, and let W denote the extended w-plane. Let $N(w, \delta)$ denote the set of all points of W at a chordal distance less than δ from w ($\delta>0$), and define a closed set $B\subset W$ as follows: $w\in B$ if and only if $w\in W$ and for any $\delta>0$ there exist $N(w_0, \delta_0)\subset N(w, \delta)$ and a component U of the preimage $f^{-1}(N(w_0, \delta_0))$ such that $f(U)$ is not dense in $N(w_0, \delta_0)$.

THEOREM 1. *Suppose that V is a domain contained in the complement of B and that U is a component of $f^{-1}(V)$. Then one of the following two statements holds:*

(i) *For any $w\in V$ there exists $\delta>0$ such that $U\cap f^{-1}(N(w, \delta))$ is relatively compact (in D).*

(ii) *For any $w\in V$ either there exists a continuous curve α: $z(t)$, $0\leqq t<1$, lying in U such that $|z(t)|\to 1$ and $f(z(t))\to w$ as $t\to 1$, or there exists $\delta>0$ such that infinitely many relatively compact components of $f^{-1}(N(w, \delta))$ are contained in U.*

The proofs of the results stated in this note will appear in a forthcoming paper.

A point $w\in W$ is called an asymptotic value of f provided there exists a continuous curve α: $z(t)$, $0\leqq t<1$, lying in D such that $|z(t)|\to 1$ and $f(z(t))\to w$ as $t\to 1$, and if in addition $z(t)\to e^{i\theta}$, then f is said to have the asymptotic value w at $e^{i\theta}$. Define a set $\Gamma_p\subset W$ as follows: $w\in\Gamma_p$ if and only if $w\in W$ and there exists $e^{i\theta}$ such that f has the asymptotic value w at $e^{i\theta}$. For any set $S\subset W$, let $A(S)=\{e^{i\theta}$: there exists $w\in S$ such that f has the asymptotic value w at $e^{i\theta}\}$.

THEOREM 2. *Suppose that $w\in B$. Then for any $\delta>0$, $A(N(w, \delta))$ has positive Lebesgue measure (in $[0, 2\pi]$) and $\Gamma_p\cap N(w, \delta)$ has positive linear measure.*

We use the definition of linear measure that is given in terms of coverings by discs. The measurability of $A(N(w, \delta))$ and Γ_p is proved in (7).

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THEOREM 3. *Suppose that V is a component of the complement of B. Then one of the following two statements holds:*

(1) *There are at most finitely many components U of $f^{-1}(V)$, each of which satisfies* (i). *In particular, f assumes as a value every point of V the same finite number of times n, multiplicities counted as usual (possibly $n=0$).*

(2) *For any $w \in V$, either w is an asymptotic value of f, or the following holds: There exists $\delta > 0$ such that $f^{-1}(N(w, \delta))$ has infinitely many relatively compact components, and in particular f assumes as a value every point of $N(w, \delta)$ infinitely many times.*

Moreover, if in case (2) *we let $n = +\infty$ (in this case each point of a residual subset of V is assumed by f infinitely many times), then any (curvilinearly) accessible boundary point of V that is assumed by f less than n times $(1 \leqq n \leqq +\infty)$, multiplicities counted as usual, is an asymptotic value of f.*

Let Γ denote the set of all asymptotic values of f. The global cluster set C and the range of values R are defined as follows: $w \in C$ if and only if $w \in W$ and there exists a sequence $\{z_n\} \subset D$ such that $|z_n| \to 1$ and $f(z_n) \to w$. $w \in R$ if and only if $w \in W$ and there exists a sequence $\{z_n\} \subset D$ such that $|z_n| \to 1$ and $f(z_n) = w$. Let int R denote the interior of R. We also define F to be the set of all $w \in W$ that satisfy the conclusion of Theorem 2. Then F is closed and by Theorem 2, $B \subset F$. Thus as a consequence of Theorem 3 we have the following result.

COROLLARY. *$C-F$ is open, and the following two equivalent inclusions hold:*

$$C - F \subset \Gamma \cup \text{int } R; \qquad C - \text{int } R \subset \Gamma \cup F.$$

Moreover, any accessible boundary point of $C-F$ that is not in R is in Γ.

Let V denote a domain contained in W, and suppose that U is a component of $f^{-1}(V)$. Define sets $R(U)$ and F_c as follows: $w \in R(U)$ if and only if $w \in V$ and $U \cap f^{-1}(\{w\})$ is infinite. $w \in F_c$ if and only if $w \in W$ and for any $\delta > 0$, $A(N(w, \delta))$ has positive Lebesgue measure (in $[0, 2\pi]$) and $\Gamma_p \cap N(w, \delta)$ contains a closed set of positive (logarithmic) capacity. Then F_c is closed and $B \subset F \subset F_c$.

THEOREM 4. *If $V \cap B = \emptyset$, then either* (i) *holds or $V - R(U)$ contains no (nondegenerate) continuum. If $V \cap F = \emptyset$, then either* (i) *holds or $V - R(U)$ has linear measure zero. If $V \cap F_c = \emptyset$, then either* (i) *holds or $V - R(U)$ contains no closed set of positive capacity.*

We conclude with a discussion of the relationship between our results and some earlier theorems. The inclusion $C-\text{int } R\subset\Gamma\cup F$ refines the inclusion $C-\text{int } R\subset\bar{\Gamma}$ given by Collingwood and Cartwright [2]. It improves their result $W-R\subset\Gamma$ under the assumption that Γ has linear measure zero, for we see that under the weaker assumption $F=\emptyset$, we have the stronger conclusion $W-\text{int } R\subset\Gamma$ (since $C-F$ is open and $C\neq\emptyset$, $C=W$ if $F=\emptyset$). If f is bounded and has radial limits of modulus one at almost all $e^{i\theta}$, then it is an immediate consequence of Theorems 2 and 3 that $B=F=\{|w|=1\}$, and we see that the corollary sharpens a theorem of Seidel [9] (see also Frostman [3]). Let us now consider the less restrictive case where f is only assumed to have an asymptotic value at each point of a set on $\{|z|=1\}$ of measure 2π. Let E be a closed subset of W such that for almost all $e^{i\theta}$, f has an asymptotic value at $e^{i\theta}$ that is in E. By Bagemihl's ambiguous-point theorem [1], $F\subset E$ (also $F_c\subset E$), and we see that our results contain some theorems of Lehto [4] and Lohwater [5]. Our results also contain theorems of Noshiro [8], Stoïlow [10], and McMillan [6].

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