CHARACTERIZATIONS OF FAVARD CLASSES FOR
FUNCTIONS OF SEVERAL VARIABLES

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1. Introduction. There are two general approaches to the study of saturation theory (for the definitions see [5]), namely the integral transform method (cf. [3], [5]) and the semigroup method (cf. [4]). In this note a third method, a distribution theoretical method will be employed, in particular to characterize the Favard (saturation) classes defined by

(1) \( V_\alpha^p = \{ f; f(x) \in L^p(E^n), |v|^\alpha f^*(v) = g^*(v), g \in L^p(E^n) \} \).

Here \( x, v \) denote vectors in \( E^n \) with \( |v| = \sqrt{v_1^2 + \cdots + v_n^2} \) and \( \alpha \) a positive parameter. \( f^*(v) \) being the Fourier transform of \( f \), this definition of \( V_\alpha^p \) is meaningful only for \( \alpha \geq 2 \). In order to extend it to \( 2 < \alpha < \infty \) we use the classes of Bessel potentials

(2) \( L_\alpha^p = \{ f; f \in L^p(E^n), (1 + |v|^{2 \alpha/2}) f^*(v) = h^*(v), h \in L^p(E^n) \} \).

This definition bears sense not only for \( 1 < \alpha \leq 2 \) but also for \( 2 < \alpha < \infty \) if the Fourier transform is taken in the distribution theoretical sense since \( (1 + |v|^{2 \alpha/2}) \) is an infinitely differentiable and slowly increasing function (in the terminology of L. Schwartz [10]). The problem is to show the equivalence of (1) and (2) for \( 1 < \alpha \leq 2 \) and to give simple characterizations of (2), e.g. in terms of differentiability properties both in the classical and the distributional (or Sobolev) sense. An equivalent definition of the classes \( L_\alpha^p \), investigated in [1], [2], [6], is given by

(3) \( L_\alpha^p = \{ f; f \in L^p(E^n), f = G_\alpha \ast h, h \in L^p(E^n) \} \),

where \( G_\alpha(x) \) is defined through \( G_\alpha(v) = (1 + |v|^\alpha)^{-\alpha/2} \), having the properties \( G_\alpha(x) \in L^1(E^n) \), \( G_\alpha(x) \equiv 0 \), \( \int_{E^n} G_\alpha(x) dx = (2\pi)^{n/2} \). \( L_\alpha^p \) is a subspace of the space of tempered distributions.

For integral values of \( \alpha \) we obtain an equivalence between \( L_\alpha^p \) and the Sobolev space

(4) \( W_\alpha^p = \left\{ f; f \in L^p(E^n), D^k f \in L^p(E^n) \text{ for every } k = (k_1, \ldots, k_n) \right\} \)

with \( k_j \geq 0 \) and \(|k| = \sum_{j=1}^{n} k_j \leq \alpha \).
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Here \( D^k f = \partial^{[k]} f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \) means the distribution derivative of \( f \).

2. Characterizations for \( 1 < p < \infty, \alpha > 0 \).

**Theorem 1.** For \( 1 < p \leq 2 \) and \( \alpha > 0 \), \( f \in V^p_\alpha \) if and only if \( f \in L^p_\alpha \).

The proof depends upon a lemma in a paper of E. M. Stein [11] where also some further characterizations of the class \( L^p_\alpha \) are given for \( 0 < \alpha < 2 \). Theorem 1 states that \( L^p_\alpha \) is a continuation of \( V^p_\alpha \) for \( p > 2 \). Thus it is sufficient to give equivalent characterizations of \( L^p_\alpha \). As a second step, using known results of A. P. Calderón [6], N. Aronszajn, F. Mulla and P. Szeptycki [2] we have

**Theorem 2.** Let \( \alpha = 1, 2, \ldots \) and \( 1 < p < \infty \). Then \( f \in L^p_\alpha \) if and only if \( f \in W^p_\alpha \).

Theorems 1, 2 enable us to prove many other characterizations of \( V^p_\alpha \) and \( L^p_\alpha \) for special values of \( \alpha \) or \( p \), especially those given in terms of ordinary derivatives. The cases \( \alpha = 1, 2 \) are the most important examples in the applications to saturation theory.

3. The case \( \alpha = 2 \). The following list of equivalences is a consequence of Theorems 1, 2 and of results of R. J. Nessel [8].

**Theorem 3.** Let \( f \in L^p(E^n), 1 < p < \infty \). The following assertions are equivalent:

(a) \( f \in V^p_2 \) (here the definition of \( V^p_2 \) is extended for \( p > 2 \) by \( |v|^2 f^\wedge (v) = g^\wedge (v), g \in L^p(E^n) \), in the distributional sense);
(b) \( f \in L^p_2 \);
(c) \( f \in W^p_2 \);
(d) \( \Delta f = g, g \in L^p(E^n) \) (\( \Delta f = \partial^2 f / \partial x_1^2 + \cdots + \partial^2 f / \partial x_n^2 \) in the distributional sense);
(e) for \( j, k = 1, 2, \ldots, n \) the functions \( f, \partial f / \partial x_j \) are absolutely continuous in each variable, and \( \partial f / \partial x_j, \partial^2 f / \partial x_j \partial x_k \in L^p(E^n) \) (the derivatives to be understood in the ordinary sense);
(f) for \( j = 1, 2, \ldots, n \) (with \( e_j = \) unit vector in \( j \)-direction)

\[
\| f(x + 2he_j) - 2f(x + he_j) + f(x) \|_p = O(h^2) \quad (h \in E^1, h \to 0);
\]

\[
\| f(x + 2u) - 2f(x + u) + f(x) \|_p = O(\| u \|^2) \quad (u \in E^n, \| u \| \to 0);
\]

\[
\| \sum f(x + jh) - 2f(x) \|_p = O(h^2) \quad (h \to 0),
\]

where the sum runs over all \( j = (j_1, \ldots, j_n) \) with \( j_k = \pm 1 \).

4. The case \( \alpha = 1 \). The Hilbert transform of a function \( f \in L^p(E^n), 1 < p < \infty \), with respect to \( x_j \) is defined by
\[ f_j^* (x) = \lim_{\varepsilon \to 0^+} \Gamma((n + 1)/2) \int_{|x-u| \leq \varepsilon} f(u) \frac{x_j - u_j}{|x-u|^{n+1}} \, du \]  \( (j = 1, 2, \ldots, n). \)

The functions \( f_j^* (x) \) form the coordinates of a vector \( (Hf)(x) = \sum_{j=1}^n e_j f_j^* (x) \) (cf. [7]) and again belong to \( L^p(E^n) \). We have, in the case \( \alpha = 1 \),

**Theorem 4.** Let \( f \in L^p(E^n) \), \( 1 < p < \infty \). The following statements are equivalent:

(a) \( f \in L^p_1 \),
(b) \( f_j^* \in L^p_1 \) for \( 1 \leq j \leq n \),
(c) \( f \in W^p_1 \),
(d) \( (\text{div } Hf)(x) = \sum_{j=1}^n (\partial / \partial x_j) f_j^*(x) \in L^p(E^n) \) (distributional derivatives),
(e) for \( j = 1, 2, \ldots, n \) the functions \( f_j^* (x) \) are absolutely continuous in each variable, and the ordinary derivatives \( \partial f_j^*/\partial x_k \) \( (k = 1, \ldots, n) \) are in \( L^p(E^n) \),
(f) for \( j, k = 1, 2, \ldots, n \),

\[ ||f_j^* (x + he_k) - f_j^*(x)|| = O(|h|) \quad (h \to 0); \]
(g) \( ||f(x+he_i) - f(x)|| = O(|h|) \quad h \to 0; \quad j = 1, 2, \ldots, n). \)

5. **The case** \( p = 2 \). For \( p = 2 \) many known equivalences between \( L^2_\alpha \) and “fractional” Sobolev spaces or various types of Besov spaces can be applied to give new characterizations of \( V^2_\alpha \) for arbitrary \( \alpha > 0 \). For instance the fractional Sobolev spaces defined for fractional \( \alpha = m + \beta \) \( (m = 0, 1, 2, \ldots; 0 < \beta < 1) \) by

\[ W^2_\alpha = \left\{ f; f \in W^2_m, \left[ \int_{E^n} \left| \frac{D_j^2 f(\cdot + u) - D_j^2 f(\cdot)}{|u|^{2\beta+n}} \right|^2 \, du \right]^{1/2} < \infty \right\} \]

and for \( \alpha = 1, 2, \ldots \) by (4) (see [9], [2, p. 74]) are equivalent to \( V^2_\alpha \) for \( \alpha > 0 \).

The present results are also connected with the theory of intermediate spaces which is presented in [4]. A detailed discussion of these results, including proofs and various extensions, will be published elsewhere.
REFERENCES


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