THE IMPOSSIBILITY OF FILLING $E^n$ WITH ARCS

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The purpose of this paper is to outline a proof of the following

**Main Theorem.** If $f$ is a closed continuous map of $E^n$ onto any space $S$, then some point in $S$ has an inverse image which is not an arc.

In 1936 J. H. Roberts [1] showed that there does not exist an upper semicontinuous (usc) collection of arcs filling the plane. Recently L. B. Treybig [2] has obtained some partial results for polygonal arcs in $E^n$. In 1955 Eldon Dyer [3] outlined a proof that there is no continuous decomposition of $E^n$ into arcs. This proof incorporates some of the ideas of both Roberts and Dyer.

We will suppose that all statements are for $E^n$ for a given $n$.

**Definitions.** If $U$ and $V$ are sets with disjoint closures, we say that an arc $\alpha$ has $k$ *folds* between $U$ and $V$ if $\alpha$ contains $k+1$ disjoint subarcs between $U$ and $V$. Furthermore, if the distance between each pair of the $k+1$ subarcs is greater than $\epsilon$, we say that the *width* of the folds is greater than $\epsilon$. If $\alpha$ contains a subarc which has endpoints in $U$ and which intersects $V$, then $\alpha$ is said to have a fold with the *bend* in $V$.

If $K$ is a set, $\epsilon>0$, let $N_\epsilon(K)$ denote the open $\epsilon$-neighborhood of $K$ in $E^n$. If $H$ is a collection of sets, let $H^*$ denote the set of all points covered by elements of $H$.

Suppose $A$ is compact and $B$ is a closed subset of $A$. If any two points of $E^n - A$ which are separated by $A$ are also separated by $B$, then $B$ is said to be *essential* in $A$. If $H$ is a usc collection of arcs and points filling $A$ and $B$ intersects each element of $H$, then $B$ is said to be *full* in $A^H$. If $B$ meets each element of $H$ in a continuum, then $B$ is said to be a *quasi-section* of $A^H$.

Assume $H$ is a usc collection of arcs and points filling the compact set $X$.

**Lemma 1.** If $Y$ is a quasi-section of $X^H$ then $Y$ is essential in $X$.

The proof is an exercise in the Vietoris mapping theorem on the Čech homologies of $X$, $Y$, and the decomposition space.

**Lemma 2.** If $K$ is full in $X^H$, $U$ is open, $\overline{U} \cap K = \emptyset$, and no element

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of $H$ has a fold between $K$ and $U$ with bend in $U$, then $X$ has an essential subset that misses $U$.

It follows from the hypothesis that for each $h \in H$, $h - (h \cap U)$ contains a unique component which intersects $K$. If $Y$ is the union of all such components, $Y$ is a quasi-section of $X^H$ and hence $Y$ is an essential subset that misses $U$.

**Remark.** Obviously, under the above conditions if $U$ is connected, $U$ cannot intersect two distinct components of $E^a = X$.

Suppose $G$ is a usc collection of arcs and points filling some complete metric space. The collection $G$ is said to be continuous at an element $g$ if for every finite chain $\mathcal{C}$ of open sets covering $g$, there exists an $\epsilon > 0$ such that each element of $G$ contained in $N_\epsilon(g)$ intersects each element of $\mathcal{C}$. The collection $G$ is said to be equicontinuous at $g$ if $G$ is continuous at $g$ and no element of $G$ contained in $N_\epsilon(g)$ contains a fold between two nonadjacent links of $\mathcal{C}$. Roberts proved that the set $G_1$ of elements at which $G$ is continuous is dense in $G$, and the set $G_2$ of elements at which $G_1$ is equicontinuous is dense in $G_1$.

Suppose $X$ is compact and $H$ is a usc collection of arcs and points filling $X$.

**Lemma 3.** If $K$ is full in $X^H$, $Q$ is a quasi-section of $X^H$ that contains an endpoint of each element of $H$, and $K \cap Q = \emptyset$, then there is a quasi-section $Y$ of $X^H$ such that $Q$ is not contained in $Y$.

Let $h_3$ denote an element of $H_3$. We can find an $\epsilon > 0$ such that $h_3$ contains no folds between $N_{2\epsilon}(K)$ and $N_\epsilon(Q)$. Since $H_1$ is equicontinuous at $h_3$, if $h_1 \in H_1$ is near $h_3$, then $h_1$ contains no folds between $N_{2\epsilon}(K)$ and $N_\epsilon(Q)$. Suppose $h \in H$, $h$ is very near $h_3$, and the component of $h - [h \cap N_\epsilon(K)]$ that meets $Q$ contains a fold between $N_{2\epsilon}(K)$ and $N_\epsilon(Q)$. This implies that every element of $H_1$ very near $h$ must also contain such a fold, since every such element must span between $K$ and $Q$, and to do this it must "follow" $h$ from $N_\epsilon(Q)$ to $N_{2\epsilon}(K)$, back to $N_\epsilon(Q)$, and again to $N_{2\epsilon}(K)$ before it can intersect $K$. Hence from Lemma 2 we have a quasi-section $Y_1$ of $X^H$ of arcs from $\text{Bd} N_\epsilon(K)$ to $Q$, and a quasi-section $Y_2$ of $Y_1$ which misses a very small open set about $h_3 \cap Q$. Trivially, $Y_2$ is a quasi-section of $X^H$, and this completes the proof of Lemma 3.

We will suppose throughout the remainder of the paper that $G$ is a usc collection of arcs filling $E^a$.

Suppose $g$ is an element at which $G$ is continuous, $U$ and $V$ are open sets with disjoint closures, and each of $U$ and $V$ contains an end-
point of \( g \). Let \( K \) denote a closed neighborhood of the endpoint of \( g \) in \( U, K \subset U \). Let \( M \) denote the set of all elements of \( G \) which intersect \( \text{Bd} \ N_\epsilon(g) \), for some small \( \epsilon \). Hence \( M^* \) is compact and if \( \epsilon \) was selected small enough, then (a) \( M^* \cap K \) is full in \( M^{G \mid M} \), and (b) \( V \) meets two components of \( E^n - M^* \). The remark following Lemma 2 implies there is an arc with a fold between \( K \cap M \) and \( V \), and hence

**Theorem 1.** There exists an element of \( G \) with a fold between \( U \) and \( V \) with the bend in \( V \).

To prove the Main Theorem, we need to find some arc with infinitely many folds between \( U \) and \( V \). It should be noted that it is insufficient to obtain a sequence \( \{ \alpha_j \}_{j=0}^\infty \) of elements of \( G \) such that each \( \alpha_j \) contains \( j \) folds between \( U \) and \( V \), since the limit of such a sequence may be an arc with no folds at all. Thus we need sequences \( \{ \alpha_j \}_{j=0}^\infty \) of arcs of \( G \) and \( \{ d_j \}_{j=0}^\infty \) of positive numbers such that for each \( j, k > j, \alpha_k \) contains \( j \) folds between \( U \) and \( V \) of width at least \( d_j \). The limit of such a sequence would be an arc with infinitely many folds between \( U \) and \( V \). The following is an analogue to a lemma of Roberts.

**Theorem 2.** There exists an open set \( W \) such that each element of \( G \) that meets \( W \) contains a fold between \( \overline{U} \) and \( \overline{V} \).

**Remark.** For \( \epsilon > 0 \) the set of all elements having a fold between \( U \) and \( V \) of width \( \geq \epsilon \) is closed. Thus using Theorem 2 and the Baire category theorem we easily obtain an open set \( W' \) and a positive number \( d \) such that each element of \( G \) that meets \( W' \) contains a fold of width greater than \( d \). The proof then proceeds similar to that of Roberts.

The proof of Theorem 2 is crucial and requires more machinery.

Note that since \( U \) and \( V \) were selected arbitrarily it is sufficient to show that for \( \epsilon > 0 \), there is an open set \( W \) such that each element that meets \( W \) contains a fold between \( N_\epsilon(U) \) and \( N_\epsilon(V) \).

From Theorem 1 there is some arc \( \alpha \) of \( G \) which contains a fold between \( U \) and \( V \) with the bend in \( V \). Let \( \alpha' \) denote a subarc of \( \alpha \) which has a fold between \( U \) and \( V \) with the bend in \( V \) but no subarc of \( \alpha' \) has this property. Hence \( \alpha' \) minus its endpoints separates \( \alpha \) into two components \( K_1 \) and \( K_2 \). Let \( U_1 \) and \( U_2 \) denote small disjoint open sets about \( K_1 \) and \( K_2 \) respectively. Thus \( \alpha \cap U \subset U_1 \cup U_2 \), and every element of \( G \) near \( \alpha \) meets either \( U_1 \) or \( U_2 \) since every such element of \( G \) must intersect \( U \) (every element near \( \alpha \) is also near the element \( g \) at which \( G \) is continuous). Hence if \( \delta \) is small enough, every element of \( G \) contained in \( N_\delta(\alpha) \) which meets both \( U_1 \) and \( U_2 \) contains a fold
between \( N_\varepsilon(U) \) and \( V \). Thus the following lemma implies Theorem 2.

Suppose \( \alpha \in G \), \( U_1 \) and \( U_2 \) are open sets, each containing an endpoint of \( \alpha \), \( \varepsilon > 0 \), and every element of \( G \) near \( \alpha \) meets either \( U_1 \) or \( U_2 \).

**Lemma 4.** There exists an open set \( W \) such that every element of \( G \) that meets \( W \) intersects both \( N_\varepsilon(U_1) \) and \( N_\varepsilon(U_2) \).

We assume this lemma to be false.

Let \( G' \) denote the usc collection of arcs and points filling \( E^n \) such that \( g' \in G' \) if and only if either (a) for some element \( g \) of \( G \), \( g' \) is a component of \( g - \{ g \cap [N_{\varepsilon/3}(U_1) \cup N_{\varepsilon/3}(U_2)] \} \) or (b) \( g' \) is a point of \( N_{\varepsilon/3}(U_1) \cup N_{\varepsilon/3}(U_2) \). Trivially, if Lemma 4 is false for \( G \) then it is false for \( G' \).

Let \( L \) denote the set of all elements of \( G' \) which are near \( \alpha \) (here \( \alpha \) is considered as a point set, since \( \alpha \in G' \)) and which fail to intersect both \( N_\varepsilon(U_1) \) and \( N_\varepsilon(U_2) \). \( L \) is naturally divided into \( L_1 \) and \( L_2 \), those elements intersecting \( N_\varepsilon(U_1) \) but not \( N_\varepsilon(U_2) \), and \( L_2 \), those elements intersecting \( N_\varepsilon(U_2) \) but not \( N_\varepsilon(U_1) \).

Let \( M \) denote \( B^n - L^* \), where \( B^n \) is a very large ball containing \( g \), \( \alpha \), \( L^* \), \( U \), \( V \), etc.

**Lemma 4.1.** \( M \) separates \( U_1 \) from \( U_2 \).

This is obvious since \( L^* = L_1^* \cup L_2^* \).

**Lemma 4.2.** \( L^* \) is dense near \( \alpha \).

This is a direct result from the assumption that Lemma 4 is false.

**Lemma 4.3.** No element of \( G' \) near \( \alpha \) contains a fold between \( N_\varepsilon(U_1) \) and \( N_\varepsilon(U_2) \).

Suppose \( \beta' \in G' \) near \( \alpha \) such that \( \beta' \) contains a fold between \( N_\varepsilon(U_1) \) and \( N_\varepsilon(U_2) \) with the bend in \( N_\varepsilon(U_2) \). Let \( \beta \in G \) such that \( \beta' \subset \beta \). If \( W \) is a very small open set about a point in the bend of \( \beta \) in \( N_\varepsilon(U_2) \), then every element of \( G \) that meets \( W \) must “follow” \( \beta \) into \( N_\varepsilon(U_1) \), since recall that every element near \( \alpha \) must intersect either \( U_1 \) or \( U_2 \). We are assuming such a \( W \) does not exist and hence this completes Lemma 4.3.

Let \( Q \) denote \( L_1^* \cap L_2^* \).

**Lemma 4.4.** Every essential subset of \( M \) contains \( Q \).

Every subset of \( M \) that fails to contain every point of \( Q \) fails to separate \( L_1^* \) from \( L_2^* \) and hence fails to separate \( U_1 \) from \( U_2 \).

**Lemma 4.5.** \( Q \) is a quasi-section of \( M^{G'\cap M} \) near \( \alpha \).
If \( g' \subset G' \), and \( p \) is a point of \( g' \cap Q \), then there is a subarc \( l_p \) of \( g' \) between \( \text{Bd} \ N_{\epsilon/2}(U_1) \) and \( \text{Bd} \ N_{\epsilon/2}(U_2) \) containing \( p \). We observe that \( l_p \) is the union of a segment in \( L_1^* \) and a segment in \( L_2^* \). Thus if \( q \) is another point of \( g' \cap Q \), since \( g' \) contains no fold between \( N_\epsilon(U_1) \) and \( N_\epsilon(U_2) \), then \( l_p = l_q \), and hence \( p \) and \( q \) lie in the same component of \( g' \cap Q \). Thus \( Q \) is a quasi-section of \( M^{G'|M} \) near \( \alpha \).

By a similar argument, if we let \( X \) denote \( L_1^* \cap M \) near \( \alpha \), then \( M' = \left[ \text{Closure of} \ (M - \text{elements of} \ G' \text{ near} \ \alpha) \right] \cup X \) is a quasi-section of \( M^{G'|M} \). Hence \( M' \) is an essential subset of \( M \) and contains \( Q \) by Lemma 4.4. It follows from the definition of \( X \) that \( X \) is compact, \( G'|X \) is a usc collection of arcs filling \( X \) such that \( \text{Bd} \ N_{\epsilon/2}(U_1) \) is full in \( X^{G'|X} \), and \( Q \) is a quasi-section of \( X \) which contains an end-point of each arc in the decomposition. Lemma 3 implies there is a quasi-section \( Y \) of \( X \) which does not contain \( Q \). However, if \( M'' = \text{cl} (M' - X) \cup Y \), \( M'' \) is a quasi-section of \( M' \) and hence a quasi-section of \( M^{G'|M} \). Thus \( M'' \) is an essential subset of \( M \) which does not contain \( Q \). This contradicts Lemma 4.4 and completes the proof of Theorem 2.

References

2. L. B. Treybig, *Concerning upper semicontinuous decompositions of \( \mathbb{R}^n \) whose nondegenerate elements are polyhedral arcs or starlike continua*, (to appear).

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